Lifshitz theories in higher spin gravity

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Review of 3-dim higher spin gravity

- Higher spin theories in $d \geq 3$ by Vasiliev et al:

$$
\begin{aligned}
dW &= W \wedge *W, \\
dB &= W * B - B * W \\
dS &= W * S - S * W \\
S * S &= C + R(B)
\end{aligned}
$$

- AdS/CFT duality to Large N vector model (such as $O(N)$ in d=3)
- Infinite tower of higher spin fields
- nonlinear coupling of matter
- nonlocal (infinite number of derivatives)
- no action, only equations of motion
- mainly classical, hard to quantize

Review of 3-dim higher spin gravity

Situation in $d = 3$ somewhat simpler:

- Infinite tower of higher spins coupled to scalar matter
- Linearizing scalar: theory can be reformulated as $hs(\lambda) \times hs(\lambda)$ Chern-Simons gauge theory
- Gaberdiel and Gopakumar: Theory dual in a 't Hooft like limit to W_N minimal model coset
- For $\lambda = \pm N$ theory truncates to $SL(N, R) \times SL(N, R)$ Chern-Simons gauge theory
- Interacting theory of spin 2,3,...,N

Review of 3-dim higher spin gravity

Chern-Simons action at level k and $-k$ and gauge group $SL(N,R)\times SL(N,R)$

$$S = S_{CS}[A] - S_{CS}[\bar{A}]$$

where

$$S_{CS}[A] = \frac{k}{4\pi}\int \mathrm{tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big).$$

For $SL(3,R)\times SL(3,R)$, theory contains spin 2 and spin 3 field

$$e_\mu = \frac{l}{2}(A_\mu - \bar{A}_\mu), \quad \omega_\mu = \frac{1}{2}(A_\mu + \bar{A}_\mu)$$

with

$$g_{\mu\nu} = \frac{1}{2}tr(e_\mu e_\nu), \quad \phi_{\mu\nu\rho} = \frac{1}{6}tr(e_{(\mu}e_\nu e_{\rho)})$$

Holographic Lifshitz space-times

- Lifshitz symmetry: anisotropic scaling wrt. space and time

$$t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}$$

- in two dimensions only one spatial coord x and time translations H , spatial translations P and Lifshitz scaling D form Lifshitz algebra

$$[P, H] = 0 \qquad [D, H] = zH \qquad [D, P] = P$$

- the energy density $\mathcal{E}$, the energy flux $\mathcal{E}^x$, the momentum density $\mathcal{P}_x$ and the stress energy tensor Π_x^x

$$\partial_t \mathcal{E} + \partial_x \mathcal{E}^x = 0, \quad \partial_t \mathcal{P}_x + \partial_x \Pi_x^x = 0, \quad z\mathcal{E} + \Pi_x^x = 0$$

Holographic Lifshitz space-times

- Holographic realization of Lifshitz

$$ds^2 = L^2\left(d\rho^2 - e^{2z\rho}dt^2 + e^{2\rho}dx^2\right)$$

Shift $\rho \to \rho + \ln\lambda$ induces Lifshitz scaling on t, x coordinates.

- For $SL(3,R) \times SL(3,R)$ CS gravity. Generators of $SL(3,R)$

$$SL(2,R): \quad L_{\pm 1}, L_0, \qquad \text{spin } 2: \quad W_{\pm 2}, W_{\pm 1}, W_0$$

- Radial gauge fixes ρ dependence with $b = exp(\rho L_0)$

$$A_\mu = b^{-1} a_\mu\, b + b^{-1}\partial_\mu b, \quad \bar{A}_\mu = b\,\bar{a}_\mu b^{-1} + b\,\partial_\mu(b^{-1})$$

Holographic Lifshitz space-times

- The following connection gives $z = 2$ Lifshitz metric.

$$a = W_2\, dt + L_1\, dx, \qquad \bar{a} = W_{-2}\, dt + L_{-1}\, dx$$

Define an asymptotic Lifshitz connection

$$A - A_{\rm Lif} \sim \mathcal{O}(1), \qquad \text{as } \rho \to \infty$$

Two goals:

- Identify the Lifshitz stress-energy complex in asymptotic Lifshitz connection
- Realize Lifshitz symmetries as residual gauge transformations on asymptotically Lifshitz connections.

Holographic Lifshitz space-times

- Asymptotic Lifshitz + flatness ($F = 0$) connection of the form

$$a_t = W_2 - 2\mathcal{L} W_0 + \frac{2}{3}\mathcal{L}' W_{-1} - 2\mathcal{W} L_{-1} + \left(\mathcal{L}^2 - \frac{1}{6}\mathcal{L}''\right) W_{-2},$$
$$a_x = L_1 - \mathcal{L} L_{-1} + \mathcal{W} W_{-2}$$

- Time evolution equation

$$\dot{\mathcal{L}} = 2\mathcal{W}', \qquad \dot{\mathcal{W}} = \frac{4}{3}(\mathcal{L}^2)' - \frac{1}{6}\mathcal{L}'''$$

Holographic Lifshitz space-times

- em-complex for Lifshitz can be identified with

$$\begin{aligned}
\mathcal{E} &= \mathcal{W} + \bar{\mathcal{W}}, \\
\mathcal{P}_x &= \mathcal{L} - \bar{\mathcal{L}}, \\
\Pi_x^{\ x} &= -2\mathcal{W} - 2\bar{\mathcal{W}}, \\
\mathcal{E}^x &= -\left(\frac{4}{3}\mathcal{L}^2 - \frac{1}{6}\partial_x^2\mathcal{L}\right) + \left(\frac{4}{3}\bar{\mathcal{L}}^2 - \frac{1}{6}\partial_x^2\bar{\mathcal{L}}\right)
\end{aligned}$$

- Where the energy density $\mathcal{E}$, the energy flux $\mathcal{E}^x$, the momentum density $\mathcal{P}_x$ and the stress energy tensor $\Pi_x^{\ x}$, with $z = 2$.

$$\partial_t\mathcal{E} + \partial_x\mathcal{E}^x = 0, \quad \partial_t\mathcal{P}_x + \partial_x\Pi_x^{\ x} = 0, \quad z\mathcal{E} + \Pi_x^{\ x} = 0$$

Holographic Lifshitz space-times

- Time evolution equation for $\mathcal{L}, W$ related to Boussinesq equation (eliminate W)

$$\ddot{\mathcal{L}} = \frac{8}{3}(\mathcal{L}^2)'' - \frac{1}{3}\mathcal{L}''''$$

- Integrable system (bi-Hamiltonian) related to W_3 algebra (as KdV is related to Virasoro)
- Infinitely many conserved commuting charges

$$\begin{aligned} q_1 &= \int dx\, \mathcal{W}, \quad q_2 = \int dx\, \mathcal{L} \\ q_3 &= \int dx\, \mathcal{W}\mathcal{L}, \quad q_4 = \int dx\, \Big(\mathcal{W}^2 + \frac{4}{9}\mathcal{L}^2 + \frac{1}{12}(\mathcal{L}')^2\Big), \cdots \end{aligned}$$

Realization of Lifshitz symmetries

- Symmetries realized as gauge transformations leaving asymptotic Lifshitz form invariant
- variation of charge

$$\delta Q(\Lambda) = -\frac{k}{2\pi}\int_{-\infty}^{\infty} dx tr(\Lambda \delta A_x)$$

if integrable leads to charge $Q(\Lambda)$

- Algebra of charges

$$\{Q(\Lambda), Q(\Gamma)\} = \delta_\Lambda Q(\Gamma)$$

Realization of Lifshitz symmetries

- Charges

$$Q(\Lambda_H) = \frac{2k}{\pi}\int_{-\infty}^{\infty} dx\mathcal{W}$$
$$Q(\Lambda_P) = \frac{2k}{\pi}\int_{-\infty}^{\infty} dx\mathcal{L}$$
$$Q(\Lambda_D) = -\frac{2k}{\pi}\int_{-\infty}^{\infty} dx(2t\mathcal{W} + x\mathcal{L})$$

- satisfy Lifshitz algebra

$$\{Q(\Lambda_H), Q(\Lambda_P)\} = 0,$$
$$\{Q(\Lambda_D), Q(\Lambda_H)\} = 2Q(\Lambda_H),$$
$$\{Q(\Lambda_D), Q(\Lambda_P)\} = Q(\Lambda_P).$$

- Open question: infinite dimensional extension ?

Higher spin black holes

- Black hole in higher spin gravity: Geometric characterization difficult as higher spin gauge transformations act on metric
- horizon, causal structure are gauge dependent
- higher spin BH for asymptotic AdS (in radial gauge)

$$a_+ = (L_1 + \mathcal{L} L_{-1} + \mathcal{W} W_{-2}), \quad a_- = \mu(W_2 + \cdots)$$

- Gauge invariant characterization of BH: holonomy around euclidean time circle is equal to BTZ, i.e. in the center of SL(3,R).

Lifshitz higher spin black holes

- For Lifshitz asymptotics replace l.c. coordinates $+ \to x, - \to t$.

$$a_x = (L_1 + \mathcal{L} L_{-1} + \mathcal{W} W_{-2}), \quad a_t = \mu_2 W_2 + \mu_1 L_1 + \cdots$$

- Nonrotating BH: $\bar{a}$ determined by a

$$\bar{a}_x = -a_x^T, \quad \bar{a}_t = a_t^T$$

- Interpretation of $\mathcal{L}, \mathcal{W}$ from em-complex, μ_1, μ_2 chemical potential.
- Role changed from AdS BH (energy $\sim \mathcal{W}$, $\beta \sim \mu_2$)

Lifshitz higher spin black holes

- Holonomy around euclidean time circle, taken the same as for BTZ

$$\mathcal{P}\exp\left(\oint_t dtA_t\right) = diag(e^{2\pi i}, 1, e^{-2\pi i})$$

- Holonomy conditions: "equation of state" relate $\mathcal{W}, \mathcal{L}$ to μ_1, μ_2.

$$0 = 3\mathcal{L}\mu_1^2 + 9\mathcal{W}\mu_1\mu_2 + 4\mathcal{L}^2\mu_2^2 - \frac{3}{4},$$
$$0 = 108\mathcal{W}^2\mu_2^3 + 8\mathcal{L}^2\mu_2\left(9\mu_1^2 - 4\mathcal{L}\mu_2^2\right) + 27\mathcal{W}\left(\mu_1^3 + 4\mathcal{L}\mu_1\mu_2^2\right).$$

Thermodynamics

- On shell CS-action I_0 reduces to boundary term.

$$I_0^{\rm os} = -4k\big(2\mathcal{L}\mu_1 + 3\mathcal{W}\mu_2\big)$$

- Add a boundary term to produce an euclidean action

$$I_1 = -8k(\mu_1\mathcal{L} + 2\mu_2\mathcal{W})$$

- Has a good variational principle

$$\delta I_1 = 8k(\mathcal{L}\delta\mu_1 + \mathcal{W}\delta\mu_2)$$

- Using the holonomy conditions

$$\frac{\partial I_1}{\partial\mu_1} = 8k\mathcal{L}, \qquad \frac{\partial I_1}{\partial\mu_2} = 8k\mathcal{W}.$$

- $\mathcal{W}, \mathcal{L}$ are charges and μ_2, μ_1 conjugate chemical potentials

Thermodynamics

- grand potential is defined in terms of partition function

$$\Phi = -\frac{1}{\beta}\ln Z = \frac{1}{\beta} I_1$$

- natural variables: temperature T and chemical potential α

$$\mu_1 = \beta\alpha = \frac{1}{T}\alpha, \qquad \mu_2 = \beta = \frac{1}{T}$$

- β multiplies a_t and hence g_{tt} and can be absorbed in periodicity of euclidean time.

Thermodynamics

- grand potential

$$\Phi = -8k\Big(\alpha\mathcal{L} + 2\mathcal{W}\Big)$$

- Thermodynamic differential

$$d\Phi = -SdT - Qd\alpha$$

- Determines charge and entropy

$$Q = -\left.\frac{\partial\Phi}{\partial\alpha}\right|_T = -8k\mathcal{L}$$
$$S = -\left.\frac{\partial\Phi}{\partial T}\right|_\alpha = \frac{1}{T}8k\Big(2\alpha\mathcal{L} + 3\mathcal{W}\Big)$$

Branches

- charges $\mathcal{L}, \mathcal{W}$: extensive. chemical potentials: μ_1, μ_2 intensive.
- Solution of the holonomy conditions:

$$
\begin{aligned}
0 &= 3\mathcal{L}\mu_1^2 + 9\mathcal{W}\mu_1\mu_2 + 4\mathcal{L}^2\mu_2^2 - \frac{3}{4}, \\
0 &= 108\mathcal{W}^2\mu_2^3 + 8\mathcal{L}^2\mu_2\left(9\mu_1^2 - 4\mathcal{L}\mu_2^2\right) + 27\mathcal{W}\left(\mu_1^3 + 4\mathcal{L}\mu_1\mu_2^2\right).
\end{aligned}
$$

has different branches: entropy, energy, grand potential all take different form.

Branches

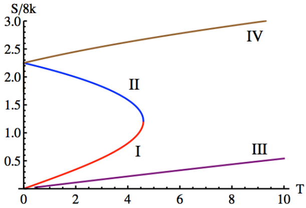
Physical conditions decide wether a branch is sensible or not

- Temperature T positive
- Entropy S positive
- S has minimum for $T \to 0$
- Local thermodynamical stability
- In metric formulation a "black hole gauge exits" which displays a regular horizon

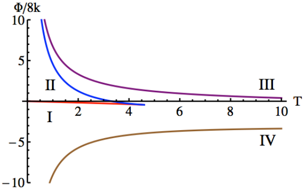
Branches: Solve $\mathcal{L}, W$ in terms of T, α has four branches or vice versa has six branches.

Branches

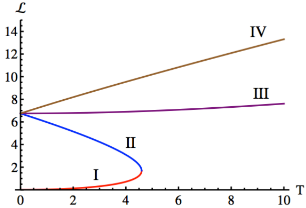
Thermodynamics as function of T for fixed α



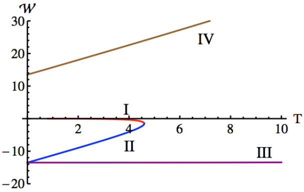
(a) Etropies.



(b) Grand potentials.



(c) $\mathcal{L}$ charge.



(d) $\mathcal{W}$ charge.

Branches

- Branch I satisfies all criteria to be sensible
- Other branches fail in one or more
- Same conclusion if we analyze branches in terms of extensive variables $\mathcal{L}, W$.

Summary

- Asymptotic Lifshitz space times: Construction of stress-energy complex and Lifshitz symmetries in the CS formulation
- Construction of BH: very similar to AdS higher spin black holes $\mu = \pm$ get replaced by t, x
- Identification of charges (energy, higher spin charge) and chemical potentials (temperature, α) different
- holonomy conditions have multiple branches, one physically sensible.

Generalizations

- CFT realization (?): $H = W_0 + \bar{W}_0$ instead of $L_0 + \bar{L}_0$.
- Infinite dimensional asymptotic symmetry generalizing Lifshitz scaling. Infinite set of commuting charges, conserved charges

$$q_3 \;=\; \int dx\, \mathcal{W}\mathcal{L}, \quad q_4 = \int dx \left(\mathcal{W}^2 + \frac{4}{9}\mathcal{L}^2 + \frac{1}{12}(\mathcal{L}')^2 \right), \cdots$$

  What about W_3 algebra, central extension ?

- Rotating solutions: Drop condition

$$\bar{a}_x \neq -a_x^T, \quad \bar{a}_t \neq a_t^T$$

  Interesting since rotating Lifshitz BH solutions have not yet been found.

Generalizations: Lifshitz theories for $hs(\lambda)$

- $hs(\lambda)$ generators V^s_m, $s = 2, 3, 4, \cdots$, $m = -s+1, -s+2, \cdots s-1$.
- Star product:

$$V^s_m * V^t_n = \frac{1}{2} \sum_{u=1,2,\cdots}^{s+t-|s-t|-1} g^{st}_u(m, n; \lambda) V^{s+t-u}_{m+n}$$

- Lifshitz space-time with $z = n$

$$a_x = V^2_1, \qquad a_t = V^{n+1}_n$$

- asymptotic Lifshitz theories with $z = 2$

$$a_x = V^2_1 + \mathcal{L} V^2_{-1} + \mathcal{W} V^3_{-2} + \mathcal{U} V^4_{-3} \cdots$$
$$a_t = a_x * a_x$$

satisfies $F = 0$ for constant $\mathcal{L}, \mathcal{W}$ etc.

Generalizations: Lifshitz theories for $hs(\lambda)$

- For $\mathcal{L}(x,t), \mathcal{W}(x,t)$ evolution equation

$$\dot{\mathcal{L}} = \frac{8-2\lambda^2}{5} W'$$
$$\dot{\mathcal{W}} = \frac{4}{3}(\mathcal{L}^2)' + \frac{27-3\lambda^2}{7} U' + \frac{1}{6}\mathcal{L}'''$$
$$\dot{\mathcal{U}} = \frac{10}{3}\mathcal{W}\mathcal{L}' - \frac{12}{5}\mathcal{L}\mathcal{W}' + \frac{64-4\lambda^2}{9}\mathcal{V}' - \frac{1}{15}\mathcal{W}''', \quad \cdots$$

  Should be related to KP hierarchy.

- Black hole ansatz

$$a_x = V_1^2 + \mathcal{L} V_{-1}^2 + \mathcal{W} V_{-2}^3 + \mathcal{U} V_{-3}^4 \cdots$$
$$a_t = \mu_1 a_x + \mu_2 a_x * a_x$$

  Holonomy condition difficult to impose.