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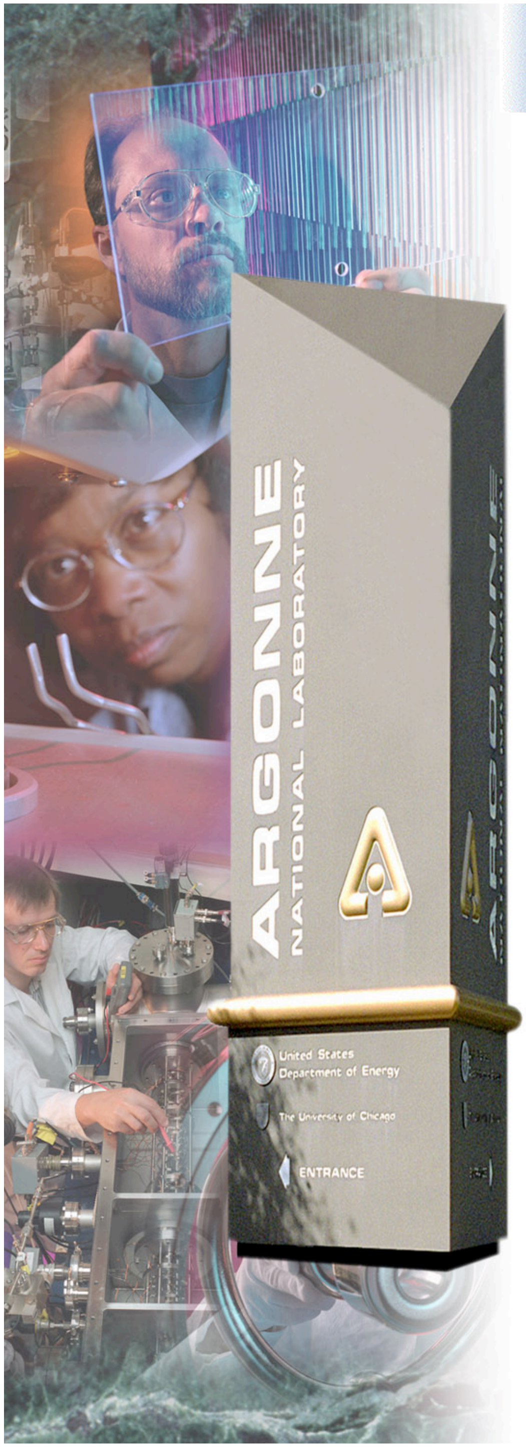


# *Quantum Effects in Gain and Start-up of Free-Electron Lasers — Wigner Function Approach*

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# Schroedinger Equation

- Wave equation:

$$i\frac{\partial \psi}{\partial \tau} = -\frac{1}{2\bar{\rho}}\frac{\partial^2 \psi}{\partial \theta^2} - i\bar{\rho}\left(Ae^{i\theta} - A^* e^{-i\theta}\right)\psi$$

- Quantum FEL parameter

$$\bar{\rho} = \frac{\rho E}{\hbar\omega}$$

- Classical limit: $\bar{\rho} >> 1$

# Wigner Distribution Function

Given a wave-field, such as coherent EM field or quantum wave field as a function of spatial coordinate $\theta$, the W-function is a function of phase space coordinate ($\theta,p$)

$$W(\theta, p,\tau)=\int_{-\infty}^{\infty} d\xi\, e^{-i\xi p}\psi^*\left(\theta - \xi/2 ,\tau\right)\psi\left(\theta + \xi/2 ,\tau\right)$$

- W is the closest object corresponding the classical phase space distribution. However, it is not a true distribution of particles since it can in general have negative values due to the fact that one cannot specify coordinate and conjugate momentum at the same time
- ***However W is useful because***:
  - **It tranforms as classical phase space distribution function**
  - **integrals of W are positive and have a physical meaning :**

    $F(\theta, \tau)=\int dp\, W(\theta,p,\tau) = |\_(\theta,\tau)|^2$: density

    $V(p, \tau)=\int d\theta\, W(\theta,p,\tau) = |\int d\theta\, e^{ip\theta} f(\theta,\tau)|^2$ :energy distribution

# Quantum "Vlasov"-Maxwell Equation

- $\overline{W}(\theta, \overline{p}, \tau)$ Satisfies (R. Bonifacio, N. Piovella,..)

$$0 = \frac{\partial \overline{W}}{\partial \tau} + \overline{p}\frac{\partial}{\partial \theta}\overline{W} \quad - \overline{\rho}\left(Ae^{i\theta} + A^* e^{-i\theta}\right)\left[\overline{W}\left(\theta, \overline{p} + \frac{1}{2\overline{\rho}}, \tau\right) - \overline{W}\left(\theta, \overline{p} - \frac{1}{2\overline{\rho}}, \tau\right)\right]$$

- For classical limit, $\overline{\rho} \to \infty$

$$\overline{\rho}\left[\overline{W}\left(\theta, \overline{p} + \frac{1}{2\overline{\rho}}, \tau\right) - \overline{W}\left(\theta, \overline{p} - \frac{1}{2\overline{\rho}}, \tau\right)\right] \qquad \to \frac{\partial W}{\partial \overline{p}}$$

- Maxwell equation

$$\left(\frac{\partial}{\partial \tau} + \frac{\nu}{2\rho}\right)A_\nu = < \int d\overline{p} e^{-i\nu\theta} W(\theta, \overline{p}, \tau) > ; \nu \approx 1$$

## Quantum Dispersion Relation

- Follow the method used in classical analysis, quantum dispersion relation for gain $\mu$:

$$\mu - \overline{\nu} - \int_{-\infty}^{\infty} \frac{\overline{\rho}\left[W_o\left(\overline{p} + \frac{1}{2\overline{\rho}}\right) - W_o\left(\overline{p} - \frac{1}{2\overline{\rho}}\right)\right]}{\overline{p} - \mu} d\overline{p} = 0$$

- Cold beam $W_o(p) = \delta(p)$

$$(\mu - \overline{\nu})\left(\mu^2 - \frac{1}{4\overline{\rho}^2}\right) - 1 = 0$$

- As noted by R. Bonifacio, et. al., this is the same as the classical DR with flat-top energy distribution.

# Wigner Function of Displaced Particles

- Wave function of reference particle and corresponding W-function

$$\Psi(\theta,\tau) \Rightarrow W(\theta, p,\tau)$$

- Wave function of a displaced particle and corresponding W-function

$$\Psi(\theta - \theta_1)\exp[ip_1\theta] \Rightarrow W(\theta - \theta_1, p - p_1)$$

- In general Wigner function of a beam of particles is a convolution of the single particle W-function and classical particle distributiondistribution

# Quantum SASE

- Wigner function for single electron at $\tau$=0

$$W^o(\theta,\eta,0)=\frac{1}{2\pi\sigma_\theta\theta_\eta}exp\left(-\frac{\theta^2}{2\sigma_\theta{}^2}-\frac{\eta^2}{2\sigma_\eta{}^2}\right)$$

$$\sigma_\theta\sigma_\eta=\frac{1}{2\overline{\rho}},\quad \overline{\rho}=\frac{\rho E}{\hbar\omega}$$

- A beam of electrons at $\tau$=0 is a collection of coherent packets each centered at a classical phase space position:

$$W(\theta,\eta,0)=\int W^o(\theta-\theta_0,\eta-\eta_0,0)F(\theta_0,\eta_0,0)d\theta_0 d\eta_0$$

$$F(\theta,\eta,0)=\frac{2\pi}{\lambda n}\sum_{j=1}^{N_e}\delta(\theta-\theta_j)\delta(\eta-\eta_j)$$

# Solve Quantum Vlasov-Maxwell Equation for SASE

- Follow the classical derivation (KJK,1986)
  Laplace transform→solve in terms of initial values→inverse Laplace Transform

$$A_\nu(\tau) = e^{\mu\tau}\left[A_\nu(0) + \int dp \frac{F_\nu(p,0)}{\mu + ip}\right]$$

$$F_\nu(p,0) = \int d\theta d\theta' dp' \exp(-i\nu\theta) W_Q(\theta - \theta', p - p') \hat{F}_{CL}(\theta', p')$$

$$\hat{F}_{CL}(\theta, p) = \frac{1}{N}\sum \delta(\theta - \theta_j)\delta(p - p_j) - \overline{F}_{CL}$$

$$< \hat{F}_{CL}(X)\hat{F}_{CL}(X') >= \delta(X - X')\overline{F}_{CL}(X)$$

$$X \equiv \{\theta, p\}$$

$\overline{F}_{CL}(X)$ is classical, smooth, average distribution

## *Quantum SASE (cont'd)*

- Follow the classical derivation (KJK, 1986):

$$|A_\nu|^2 = e^{2\tau} e^{-\sigma_\theta^2} \int d\eta V(\eta) \left| \frac{1}{\sqrt{2\pi}\sigma_\eta} \int d\eta' \frac{\exp[-\frac{1}{2}\frac{(\eta'-\eta)^2}{\sigma_\eta^2}}{(\mu - i\eta')} \right|^2$$

$$V(\eta) = \frac{1}{\sqrt{2\pi}\sigma_{c\eta}} e^{-\frac{\eta^2}{2\sigma_{c\eta}^2}}$$

- Compare this with the classical expression:

$$\left|A^{CL}{}_\nu\right|^2 = e^{2\tau} \int d\eta \frac{V(\eta)}{|\mu - i\eta|^2}$$