A NEW 3D MHD ALGORITHM: THE DISTRIBUTION FUNCTION METHOD

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A novel 3D MHD algorithm is developed based upon a distribution function method. The 3D MHD equations are written in conservative form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{1}$$

$$\frac{\partial \rho \mathbf{V}}{\partial t} + \nabla \cdot \left(\rho \mathbf{V}\mathbf{V} + \left(P + \frac{B^2}{8\pi} \right) \underline{\mathbf{I}} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) = 0 \tag{2}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \left(\mathbf{V} \left(\epsilon + P + \frac{B^2}{8\pi} \right) - \frac{\mathbf{B}}{4\pi} (\mathbf{V} \cdot \mathbf{B}) \right) = 0 \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \tag{4}$$

where

$$\epsilon = \frac{1}{2} \rho V^2 + \frac{3P}{2} + \frac{B^2}{8\pi} \quad . \tag{5}$$

The equations are closed by assuming an adiabatic equation of state: P/ρ^γ = constant.

The MHD equations (1) – (3) are integrated over a cell volume $dx\, dy\, dz$ and written as

$$\frac{\partial \rho_T}{\partial t} = - \oint \rho \mathbf{V} \cdot d\mathbf{A} = - \oint \mathcal{F}_\rho \cdot d\mathbf{A} \tag{6}$$

$$\frac{\partial (\rho \mathbf{V})_T}{\partial t} = - \oint \left(\rho \mathbf{V}\mathbf{V} + \left(P + \frac{B^2}{8\pi} \right) \underline{\mathbf{I}} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) \cdot d\mathbf{A} = - \oint \underline{\mathcal{F}}_{\rho V} \cdot d\mathbf{A} \tag{7}$$

$$\frac{\partial \epsilon_T}{\partial t} = - \oint \left(\mathbf{V} \left(\epsilon + P + \frac{B^2}{8\pi} \right) - \frac{\mathbf{B}}{4\pi} (\mathbf{V} \cdot \mathbf{B}) \right) \cdot d\mathbf{A} = - \oint \mathcal{F}_\epsilon \cdot d\mathbf{A} \tag{8}$$

where $\mathcal{F}_f$ is the flux of f, the subscript T denotes the total mass, momentum, and energy in a cell, the integral sign $\oint$ denotes a summation over the cell faces that define a volume element $dxdydz$, and $d\mathbf{A} = dydz\, N_x\, \hat{\mathbf{e}}_x + dxdz\, N_y\, \hat{\mathbf{e}}_y + dxdy\, N_a\, \hat{\mathbf{e}}_z$ where $\mathbf{N}$ is the unit normal and is taken to be ‘outward’ from the cell face. The total mass, momentum, and energy at time $t + \Delta t$ is then

$$\rho_T^{t+\Delta t} = \rho_T^t - \Delta t \oint \mathcal{F}_\rho^{t+\Delta t/2} \cdot d\mathbf{A} \tag{9}$$

$$(\rho \mathbf{V})_T^{t+\Delta t} = (\rho \mathbf{V})_T^t - \Delta t \oint \mathcal{F}_{\rho V}^{t+\Delta t/2} \cdot d\mathbf{A} \tag{10}$$

$$\epsilon_T^{t+\Delta t} = \epsilon_T^t - \Delta t \oint \mathcal{F}_\epsilon^{t+\Delta t/2} \cdot d\mathbf{A} \tag{11}$$

where the fluxes are calculated at the half-time step. The new feature of the algorithm is that the fluxes $\mathcal{F}$ are calculated by integrating an appropriate distribution function over velocity space. The distribution function is determined as follows.

Consider the generalized transport equation

$$\frac{\partial n\left\langle \chi \right\rangle}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot n\left\langle \chi \mathbf{v} \right\rangle - n\left\langle \mathbf{a} \cdot \frac{\partial \chi}{\partial \mathbf{v}} \right\rangle = 0 \tag{12}$$

where

$$\mathbf{a} = \frac{\nabla \mathbf{M}}{8\pi\rho}; \quad \mathbf{M} = B_x^2\,\hat{\mathbf{e}}_x + B_y^2\,\hat{\mathbf{e}}_y + B_z^2\,\hat{\mathbf{e}}_z \tag{13}$$

and the angular brackets are defined by $\langle \mathcal{G} \rangle = \int d\mathbf{v}\,\mathcal{G}(\mathbf{v})\,F(\mathbf{v})$ with $1 = \int d\mathbf{v}\,F(\mathbf{v})$. The continuity equation is obtained from (12) by setting $\chi = m$, the momentum equation by setting $\chi = m\mathbf{v}$, and the energy equation by setting $\chi = mv^2/2$. The question to be answered is the following: Can a distribution function $F(\mathbf{v})$ be found such that, when it is substituted into the transport equation (12), the ideal MHD equations (1) – (3) are recovered?

The following distribution function satisfies this requirement

$$F = F_1 + F_2 \tag{14}$$

where

$$F_1 = \frac{\exp(-u_1^2)}{\left(\pi v_{1x}^2\right)^{1/2}} \frac{\exp(-v_1^2)}{\left(\pi v_{1y}^2\right)^{1/2}} \frac{\exp(-w_1^2)}{\left(\pi v_{1z}^2\right)^{1/2}} \tag{15}$$

with $u_1 = (v_x - V_x)/v_{1x}$, $v_1 = (v_y - V_y)/v_{1y}$, $w_1 = (v_z - V_z)/v_{1z}$, $v_{1x}^2 = 2C_s^2/\gamma + V_{Ay}^2 + V_{Az}^2$, $v_{1y}^2 = 2C_s^2/\gamma + V_{Ax}^2 + V_{Az}^2$, $v_{1z}^2 = 2C_s^2/\gamma + V_{Ax}^2 + V_{Ay}^2$, $C_s^2 = \gamma P/\rho$, and

$$F_2 = -\left(u_2 v_2 + u_2 w_2 + v_2 w_2\right) \frac{\exp(-u_2^2)}{\left(\pi v_{2x}^2\right)^{1/2}} \frac{\exp(-v_2^2)}{\left(\pi v_{2y}^2\right)^{1/2}} \frac{\exp(-w_2^2)}{\left(\pi v_{2z}^2\right)^{1/2}} \tag{16}$$

with $u_2 = (v_x - V_x)/v_{1x}$, $v_2 = (v_y - V_y)/v_{2y}$, $w_2 = (v_z - V_z)/v_{2z}$, $v_{2x}^2 = 4V_{Ax}^2$, $v_{2y}^2 = 4V_{Ay}^2$, $v_{2z}^2 = 4V_{Az}^2$. The MHD equations (1) – (3) can be recovered from the transport equation (12) using the distributions defined by (15) and (16).

As a specific example of the method, consider the continuity equation which is written as

$$\frac{\partial \rho_T}{\partial t} = -\sum \mathcal{F}_{\rho x} dA_x - \sum \mathcal{F}_{\rho y} dA_y - \sum \mathcal{F}_{\rho z} dA_z \tag{17}$$

where $\mathcal{F}_\rho = \rho\mathbf{V}$, and the summation is over the two cell faces in the x, y, and z components. The variables ρ and $\mathbf{V}$ can be discontinuous across a cell face; in order to calculate the total flux across an interface, 'left' and 'right' side values are needed of ρ and $\mathbf{V}$. These are obtained using a high order interpolation scheme (*Lyon*, private communication) and a flux limiter (e.g., partial donor cell method (*Hain*, 1987)). The flux in the x-direction at an interface $i + 1/2$ is calculated as follows

$$\mathcal{F}_{\rho,x,i+1/2} = \mathcal{F}_{L,\rho,x,i+1/2} + \mathcal{F}_{R,\rho,x,i+1/2} \tag{18}$$

where

$$\mathcal{F}_{L,\rho,x,i+1/2} = \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \int_{0}^{\infty} dv_x\, v_x\, \rho_L F_L \tag{19}$$

$$\mathcal{F}_{R,\rho,x,i+1/2} = \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \int_{-\infty}^{0} dv_x\, v_x \rho_R F_R \tag{20}$$

where the subscripts L and R indicate that the variables used in F correspond to 'left-side' and 'right-side' states. The first term on the right-hand-side of (19) transports mass across the interface from the 'left-side'; only particles moving in the positive x-direction contribute to the flux so the integration limits on v_x are from 0 to ∞. The second term on the right-hand-side of (19) transports mass across the interface from the 'right-side'; only particles moving in the negative x-direction contribute to the flux so the integration limits on v_x are from -∞ to 0. The sum of these two fluxes then gives the total flux across the interface $i+1/2$. The mass flux in the x-direction at interface $i-1/2$ is calculated the same way. The mass fluxes in the y- and z-directions are calculated similarly. The same method is used to update the momentum and energy equations.

This technique is also used to update the magnetic field. The evolution of the magnetic field is governed by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) = -c\nabla \times \mathbf{E} \quad . \tag{21}$$

The normal component of the field is integrated over a cell face

$$\int d\mathbf{A}\cdot \frac{\partial \mathbf{B}}{\partial t} = -c\int d\mathbf{A}\cdot \nabla \times \mathbf{E} = -c\oint \mathbf{E}\cdot d\mathbf{l} \tag{22}$$

where $\oint$ denotes a line integral around the cell face. The electric field is

$$\mathbf{E} = -\frac{1}{c}\left(V_y B_z - V_z B_y\right)\hat{\mathbf{e}}_x + \frac{1}{c}\left(V_z B_x - V_z B_x\right)\hat{\mathbf{e}}_y - \frac{1}{c}\left(V_x B_y - V_y B_x\right)\hat{\mathbf{e}}_z \quad . \tag{23}$$

As an example, the x-flux through each interface is

$$\frac{\partial \Phi_x}{\partial t} = -c\sum E_y\, dy - c\sum E_z\, dz \tag{24}$$

where $\Phi_x = B_x\, dydz$ and the summation is over the two faces associated with each cell. Thus, the updated x-flux is

$$\Phi_x^{t+\Delta t} = \Phi_x^t + c\Delta t\left(\sum E_y\, dy + \sum E_z\, dz\right)^{t+\Delta t/2} \quad . \tag{25}$$

The distribution method is used to calculate the electric field. For example, the x component of the electric field is found from

$$E_x = -\frac{1}{c}\left(V_y B_z - V_z B_y\right) = -\frac{1}{c}\left(\int F(v) v_y B_z\, d^3v - \int F(v) v_z B_y\, d^3v\right)$$

where the velocity integration is done appropriately. These integrations are more complicated than those associated with the hydrodynamic variables, but are straightforward nevertheless.

As an example of this technique we present results for the Brio-Wu shock problem (*Brio and Wu*, 1988). This problem is similar to the Sod shock problem (*Sod*, 1978) except that both tangential and normal magnetic fields are included. The density n, magnetic field B_y, and the velocities V_x and V_y profiles are shown in Fig. 1. The mesh is 800 cells and the initial conditions are the following: $\rho_R = 0.12, \rho_L = 0.96, P_R = 0.1, P_L = 1, B_{yR} = -0.8, B_{yL} = 0.8, B_x = 0.75 B_{yL}$, and $\gamma = 5/3$. The solid curves are the initial conditions. The important features of Fig. 1 are that the fast rarefaction waves (FR), slow compound wave (SM), contact discontinuity (CD), and slow shock (SS) are well resolved.

A nice feature of the distribution function method is that it can be used to solved more general forms of the MHD equations. For example, finite Larmor radius (FLR) effects and temperature anisotropies (e.g., $T_\perp$ and $T_\parallel$) can be included by choosing appropriate distribution functions. The 2D MHD FLR equations have been solved using this technique (*Huba*, 1996).

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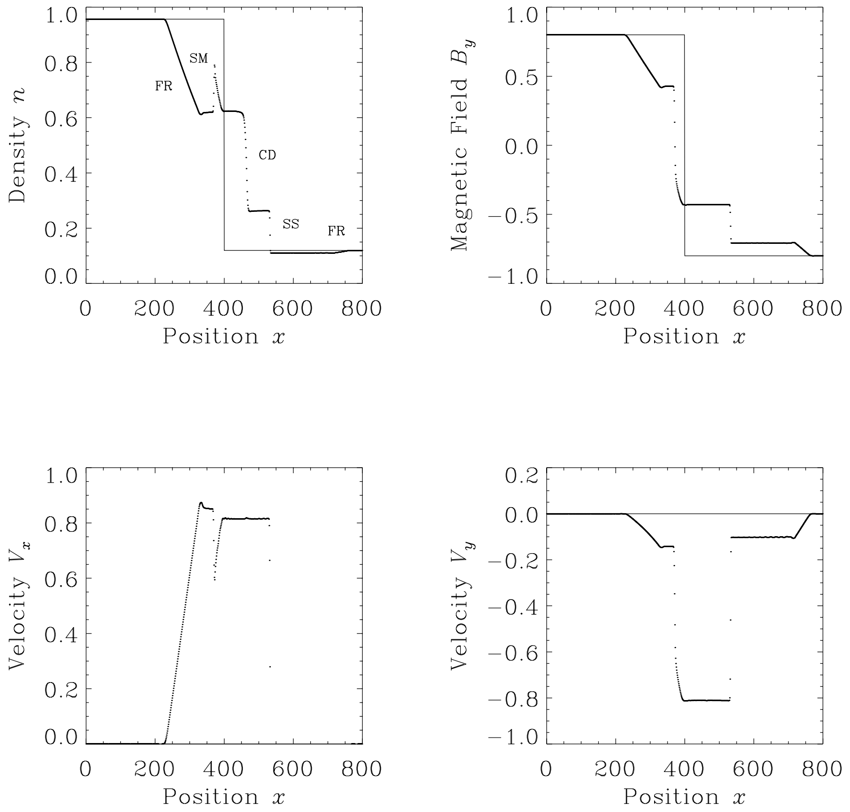

Figure 1

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