Helicities of Electron Magnetohydrodynamic Currents and Fields in Plasmas

R.L. Stenzel, J.M. Urrutia, and C.L. Rousculp
Department of Physics, University of California, Los Angeles, California 90024-1547
(Received 16 March 1994; revised manuscript received 14 September 1994)

Transient plasma currents are excited with pulsed electrodes in a large laboratory plasma in a parameter regime characterized by electron magnetohydrodynamics \( \omega_{e}\sim 1/\tau_{\text{pulse}} \). The combination of Hall currents and field-aligned currents gives rise to three-dimensional vortices whose helicity is manifested by linked and knotted current and field lines. The relation between magnetic helicity, \( \mathbf{j} \cdot \mathbf{B} \), current helicity, \( \mathbf{J} \cdot \mathbf{B} \), and their transport properties are investigated.

PACS numbers: 52.35.Hr, 52.30.Bt, 52.40.Db, 52.70.Ds

The topology of magnetic fields in plasmas is a topic of fundamental importance in fusion and space plasmas. While the attention is commonly focused on magnetohydrodynamic (MHD) plasmas [1], more recently the regime of electron MHD (EMHD, i.e., magnetized electrons, unmagnetized ions) has become of interest with its many applications such as plasma opening switches, electrodynamics tethers, and plasma releases in space [2–5].

In EMHD, Ohm’s law is dominated by electron Hall currents. The latter close like the fields of a short solenoid, such that the resultant topology is that of a 3D vortex (Hill’s vortex, spheromak [10]). Trefoil knots are observed in lines of \( \mathbf{A} \), \( \mathbf{B} \), and \( \mathbf{J} \). During the pulse propagation, both energy and helicity decay at the same rate, but the normalized helicity (e.g., \( \mathbf{J} \cdot \mathbf{B} / J B \)) remains constant. Hence, the topology does not change. Helicity transport studies show that magnetic helicity is volume generated, while current helicity is injected by the electrode. Dissipation is by collisions rather than Landau damping as in helicons [11]. On time scales short compared to diffusion times, the magnetic field is shown to be frozen to the electron fluid (\( \mathbf{B} \propto \mathbf{j} \times \mathbf{v}_e \)). The electromagnetic fields exert essentially no force on the electron fluid, \( \mathbf{f} = n_e e \mathbf{E} + \mathbf{J} \times (\mathbf{B} + \mathbf{b}_0) \).}

The experimental setup sketched in Fig. 1 consists of a large (1 m diam \( \times 2.5 \) m length) afterglow plasma column \((n_e = 9 \times 10^{10} \text{ cm}^{-3}, k_B T_e = 0.8 \text{ eV}, B_0 = 10 \text{ G}, \text{Ar}, p_n = 3 \times 10^{-4} \text{ Torr})\) into which a disk electrode \((4 \text{ cm diam})\) with an insulated shaft is axially inserted. A positive current pulse \((I_{\text{pulse}} = 150 \text{ ns}, I_{\text{max}} = 50 \text{ mA})\) is applied using the chamber wall as the ion collecting return electrode. With a triple magnetic probe \((1 \text{ cm diam})\) movable in three dimensions, the time-dependent magnetic field \((\mathbf{B}_x, \mathbf{B}_y, \mathbf{B}_z)\) is recorded from repeated experiments in the central plasma volume \((26 \text{ cm} \perp \mathbf{B}_0, 55 \text{ cm} \parallel \mathbf{B}_0)\) with \(\Delta t = 10 \text{ ns} \) time resolution and digitally stored. The current density is calculated from Ampère’s law, \( \mathbf{J}(r, t) = \nabla \times \mathbf{B}(r, t)/\mu_0 \), where \( \mathbf{J} \) includes the negligible vacuum displacement current \( (\epsilon_0 \partial \mathbf{E}/\partial t) / |\mathbf{J}| = \epsilon_0 \omega_0 \mathbf{E}/\sigma H \mathbf{E} = \omega_0 \epsilon_0 \omega_0 / \omega_0 \approx 10^{-5} \), when Hall currents dominate as shown below). The current penetrates from the electrode into the plasma along \( \mathbf{B}_0 \) at a charac-

![FIG. 1. Experimental setup, parameters, and pulsed currents in a magnetoplasma.](image-url)
For the EMHD regime, the Hall term is dominant, followed by the inertial and resistive terms \( \omega_{ce} = eB_0/m_e \gg \partial/\partial t \gg \nu \). Neglecting collisions, Faraday’s law yields
\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\frac{m_e}{n_e e^2} \left( \omega_{ce} \cdot \nabla \mathbf{J} + \nabla \times \frac{\partial \mathbf{J}}{\partial t} \right),
\]
which, for \( \omega_{ce} \cdot \nabla = \omega_{ce} \partial/\partial z \approx -\omega_{ce}/v \partial/\partial t \), can be integrated to obtain
\[
\mathbf{B} = \mathbf{J}/\sigma_H - \nabla \times \mathbf{J}/\sigma_H \omega_{ce}.
\]
Since \( \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{B} = 0 \), inspection of Eq. (3) yields
\[
\mathbf{A} \approx \mathbf{B}/\mu_0 \omega_{ce} - \mathbf{J}/\sigma_H \omega_{ce}.
\]
where \( \sigma_H = n_e e/B_0 \) is the Hall conductivity. Hence, to first order \( \mathbf{A}(r) \approx \mathbf{B}(r) \approx \mathbf{J}(r) \).

The helicity density of the \( \mathbf{J} \) field, \( h = \mathbf{J} \cdot \mathbf{B}(x = 0, y, z; t = 0.3 \mu s) \), shown in Fig. 3(a), exhibits a large positive pulse in the center, followed by smaller secondary pulses, which are induced by the primary vortex [also visible in Fig. 2(a)]. The magnetic helicity density, \( \mathbf{A} \cdot \mathbf{B} \), has a similar distribution. When integrated over the transverse cross section, \( \iint \mathbf{J} \cdot \mathbf{B} \, dx \, dy \), axial propagation and decay of helicity can be observed for the different pulses in a \( z-t \) diagram [Fig. 3(b)]. The first pulse carries most of the helicity, propagates at \( v_0 \approx 1.4 \times 10^6 \text{ cm/s} \), and decays within the volume. The typical amplitude decays along the propagation characteristics (line between diamonds, \( z = v_0 t \)) is shown in Fig. 3(c). Magnetic energy (dashed line), current helicity (solid line), and magnetic helicity (not shown for clarity) decay at the same rate, since \( \mathbf{A}(r) \approx \mathbf{B}(r) \approx \mathbf{J}(r) \), as shown above. However, the suitably normalized helicity, \( \iint \mathbf{J} \cdot \mathbf{B} \, dx \, dy/ \iint \mathbf{B} \, dx \, dy \), is constant and close to unity, indicating that the average angle between the vectors does not change. Thus, the topology does not change as the helicity decays.

Finally, integration along \( z \) of the data presented in Fig. 3(b) yields the total helicity in the measurement volume, \( H = \int h \, dV \). This is shown in Fig. 4(a) for the magnetic field helicity and in Fig. 4(b) for the current density helicity (or electron fluid velocity). The temporal growth and decay of helicity during the passage of pulses through the volume can be analyzed for the helicities analogous to that of the energy, i.e., through Poynting’s theorem (\( \partial U_{en}/\partial t = -\int_S \mathbf{S} \cdot \, da - P_{\text{dissip}} \)), where \( \mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0 \). The time rate of change of the magnetic helicity can be expressed as
\[
\dot{H}_B = -\iint_S (\mathbf{E}_{\text{ind}} \times \mathbf{A} + 2 \phi \mathbf{B}) \cdot \, da - 2 \iint \mathbf{E} \cdot \mathbf{B} \, dV,
\]
where \( \phi \) is the scalar potential [1.6]. The first integral describes the flux of magnetic helicity across closed surfaces, which consists of “ac” helicity injection \( \iint_S (\partial \mathbf{A}/\partial t \times \mathbf{A}) \cdot \, da \) and “dc” helicity injection

FIG. 2. Field topology of EMHD current pulses. (a) Snapshot of the vector potential field, \( \mathbf{A}(r, t = 0.3 \mu s) \), displayed as unit vectors in two orthogonal planes (third coordinate indicated by adjacent diamonds). The toroidal field in the transverse plane is due to Hall currents. The solenoidal field in the center of the \( y-z \) plane is due to field-aligned currents excited earlier by the electrode. Both fields are linked to form a 3D vortex. A similar topology exists for \( \mathbf{B} = \nabla \times \mathbf{A} \) and \( \mathbf{J} = \nabla \times \mathbf{B}/\mu_0 \). (b) Typical field lines of \( \mathbf{A}, \mathbf{B}, \) and \( \mathbf{J} \) in the main vortex (\( z = 30 \) cm) displayed as “sheets” (two adjacent lines connected by a surface) or “tubes” (four lines started on the corners of a square, connected by sheets). The fields are internally twisted, knotted, writhed, and linked with other lines (not shown for clarity), i.e., they exhibit helicity.

**References**

[1.6]
FIG. 3. Helicity density of the vector field $\mathbf{J} = -n_e \mathbf{v}_e$, showing positive helicity for all current pulses: the primary in the center and induced, secondary ones on the left. The pulses propagate at $v_{\text{whistler}} = 1.4 \times 10^6 \text{ cm/s along } z$ (|| $\mathbf{B}_0$) (3 dB = factor of 2 per contour; ±8 contours; zero contour has double width). (b) Helicity density integrated over transverse cross section showing propagation, damping, and relative helicities of the three pulses. Variation of integration limits (not shown) indicate that damping is not due to transverse outflow. (c) Helicity amplitude of the first major pulse along its propagation characteristics, $z = v_q t'$ [straight line between diamonds in (b)]. For comparison, the magnetic energy (dashed line) along the same characteristic is shown and found to exhibit a similar decay. However, the normalized helicity remains constant and close to unity, implying that $\mathbf{J}$ and $\mathbf{B}$ are, on the average, well aligned and their topologies do not change due to dissipation.

$-2 \int_S \phi \mathbf{B} \cdot d\mathbf{a}$. The second integral describes helicity generation and losses within the volume. These rates have been time integrated and displayed in Fig. 4(a). The ac-injected helicity is found to be negligible. The dc flux, $-2 \int_S \phi \mathbf{B} \cdot d\mathbf{a}$, exhibits both injection and outflow when the pulses enter and leave the volume. Most of the helicity is accounted for by volume production and injection, mainly by the dc term, $-2 \int_S (\mathbf{E} \times \mathbf{J}) \cdot d\mathbf{a} = \int_S \phi \mathbf{B} \cdot d\mathbf{a}$, and lost by volume dissipation, $-2 \int_S \mathbf{E} \cdot (\nabla \times \mathbf{J}) dV < 0$. The transport resembles that of the energy since $\mathbf{E} \times \mathbf{J} < \mathbf{E} \times \mathbf{H} = \mathbf{S}$ (Poynting's vector) and $\mathbf{E} \cdot (\nabla \times \mathbf{J}) \propto \mathbf{E} \cdot \mathbf{J}$.  

$-2 \int_S \phi \mathbf{B} \cdot d\mathbf{a}$, but the experimental observation is $> 0$. The breakdown of Eq. (3) is due to the boundary conditions of $\mathbf{B}$ and $\mathbf{J}$ at the electrode ($\mathbf{B} \perp \mathbf{J}$).
The transport equation for $H_J$ is given by

$$\dot{H}_J = \int \left[ B \cdot \left( \nabla \times \frac{\partial B}{\partial t} \right) \right] / \mu_0 - (\nabla \times E) \cdot J \, dV$$

$$= \int \left( \frac{\partial B}{\partial t} \times B / \mu_0 - 2E \times J \right) \cdot da - 2$$

$$\times \int E \cdot (\nabla \times J) \, dV,$$

(7)

where the helicity “Poynting’s” vector has again an ac injection term, $\int_S (\partial B/\partial t \times B/\mu_0) \cdot da$, and a dc term, $-2 \int_S (E \times J) \cdot da$. Figure 4(b) shows the helicity contributions produced by each term. In contrast to the magnetic helicity, there is no volume generation but only volume dissipation. All the current helicity is injected, $-2 \int_S (E \times J) \cdot da = 2 \iint E_x J_y dx \, dy = 2 \sigma_H \iint E_x^2 dx \, dy > 0$, where $J_y$ is derived from Eq. (1) by neglecting the inertial and resistive terms. This expression holds in the plasma but not at the surface of the electrode. Thus, the helicity flows around the electrode similar to Poynting’s vector, $S$.

Thus, in a Hall current-dominated regime ($|J| / |\nabla \times J| \gg v_H/\omega_e$), one has (i) $A \times B \approx J$ [Fig. 2(b)], hence $A \cdot B > 0$, $J \cdot B > 0$ [Fig. 3(a)]; (ii) normalized helicities constant and near unity [Fig. 3(c)]; (iii) constant helicity ratio, $J \cdot B / A \cdot B \approx \mu_0 (\sigma_H v_H)^2$; (iv) helicity proportional to magnetic energy, $J \cdot B = 2L^{-1}(B^2/2\mu_0)$, where $L = (\mu_0 \sigma_H v_H)^{-1}$; (v) ac helicity injection of $J \cdot B$ proportional to the Poynting’s flux based on the small inductive electric field, $\int_S (\partial B/\partial t \times B/\mu_0) \cdot da = L^{-1} \int_S (E_{ind} \times H) \cdot da$; (vi) dc helicity injection of $J \cdot B$ proportional to the Poynting’s flux, $2 \int_S (E \times J) \cdot da = L^{-1} \int_S (E \times H) \cdot da$; and (vii) the volume term, $\int E \cdot (\nabla \times J) \, dV = L^{-1} \int \left( E \cdot J \right) \, dV$, whose time-integrated asymptotic value ($t \rightarrow 1 \mu s$) describes the total current helicity/energy dissipated. These simplified relations explain the observations qualitatively, but corrections for inertial and Ohmic terms improve quantitative comparisons.

In conclusion, first direct measurements of the helicity properties of a three-dimensional EMHD current system produced by a simple geometry have been presented. Helicity arises self-consistently from the linkage of Hall currents with field-aligned currents. The former are always produced by the latter since the immobile ions cannot neutralize the electron charge imbalances. The relation between helicities and their transport have been analyzed and explained by basic equations. These results are of intrinsic interest and of possible relevance to various EMHD applications, theories, and simulations.

The authors gratefully acknowledge support from Grants No. NSF PHY93-03821 and No. NASA NAGW-1570.