Space and mathematics
Two small examples

Claus Lämmerzahl

thanks to Hansjörg Dittus, Wolfgang Fischer, Eva Hackmann, Valeria Kagramanova, Jutta Kunz, Peter Richter, Silvia Scheithauer

Centre for Applied Space Technology and Microgravity (ZARM),
University of Bremen, 20359 Bremen, Germany

Quantum Engineering and Space–Time Research (QUEST)

From Quantum to Cosmos 3
Airlie, July 6 – 10, 2008

Motivation

- Astronomy is the origin of all mathematics
- Led to huge number of new results
- This extends to space physics

Here are two small examples for mathematics motivated by problems in space physics:

1. The analytical solution of the geodesic equation in a Schwarzschild–(anti-)de Sitter space-time
2. The analytical solution of a deformation of a cavity in a gravity gradient field

- Kepler problem
- stability
- dynamical systems
- many body problems
- KAM theory
- Potential theory
- Relativistic equations of motion
- Einstein equations
- AdS/CFT correspondence

Outline

- Part I: Geodesic equation in Schwarzschild–de Sitter space–times
- Part II: Deformation of solids in gravitational fields

Outline

- Part I: Geodesic equation in Schwarzschild–de Sitter space–times
- Part II: Deformation of solids in gravitational fields

Outline of Part I

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. Summary

Outline

- Part I: Geodesic equation in Schwarzschild–de Sitter space–times
- Part II: Deformation of solids in gravitational fields

Outline of Part II

5. The problem
6. The model
7. Basic equation and boundary conditions
8. Analytical solution
9. Application
10. Summary

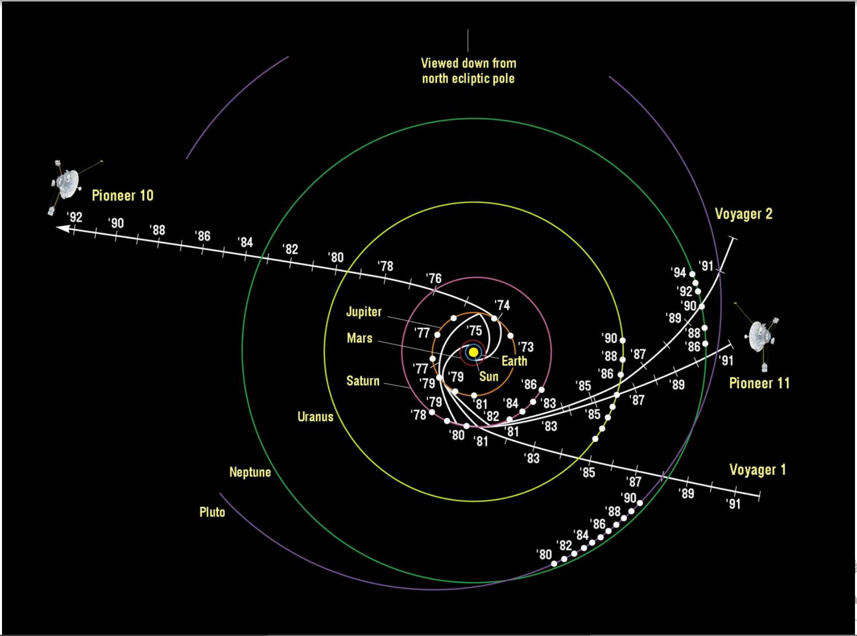
Part I

Geodesic equation in Schwarzschild–de Sitter space–times

Outline

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. Summary

Pioneer anomaly



Pioneer anomaly

The observation (Anderson et al 1998, 2002)

- Measured anomalous, uniformly blue shifted Doppler frequency drift

$$\frac{d\nu}{dt} = (5.99 \pm 0.01) \cdot 10^{-9}\ \mathrm{Hz/s}$$

- Can be interpreted as a constant acceleration

$$\boxed{a_{\mathrm{Pioneer}} = (8.74 \pm 1.33) \cdot 10^{-10}\ \mathrm{m/s^2}}$$

- Acceleration constant and toward the Sun
- Temporal and spatial variations less than 3%

Question

- Mystery: $a_{\mathrm{Pioneer}} \sim a_{\mathrm{MOND}} \sim c\,H$
- Is the Pioneer anomaly of cosmological origin?
- Is the Pioneer anomaly due to a cosmological constant?

Pioneer anomaly

The observation (Anderson et al 1998, 2002)

- Measured anomalous, uniformly blue shifted Doppler frequency drift

$$\frac{d\nu}{dt} = (5.99 \pm 0.01) \cdot 10^{-9}\ \mathrm{Hz/s}$$

- Can be interpreted as a constant acceleration

$$\boxed{a_{\mathrm{Pioneer}} = (8.74 \pm 1.33) \cdot 10^{-10}\ \mathrm{m/s^2}}$$

- Acceleration constant and toward the Sun
- Temporal and spatial variations less than 3%

Question

- Mystery: $a_{\mathrm{Pioneer}} \sim a_{\mathrm{MOND}} \sim c\,H$
- Is the Pioneer anomaly of cosmological origin?
- Is the Pioneer anomaly due to a cosmological constant?

Schwarzschild–(anti-)de Sitter space–time

- First order, post–Newtonian approximation:
  Kagramanova, Kunz & C.L., *Phys. Lett.* **B 634**, 465 (2006)
  Kerr, Hauck & Mashhoon, *Class. Quantum Grav.* **20**, 2727 (2003)
  Sereno & Jetzer, *Phys. Rev.* **D 73**, 063004 (2006)

The metric

- The metric

$$ds^2 = \alpha dt^2 - \alpha^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) ,$$

with

$$\alpha = 1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2$$

and Λ is the cosmological constant and M the mass of the source.

- $\Lambda < 0$ attraction, $\Lambda > 0$ repulsion

Schwarzschild–(anti-)de Sitter first order effects

Results

Observed effect	Estimate on Λ
gravitational redshift	$\lvert\Lambda\rvert \leq 10^{-27}\ \mathrm{m}^{-2}$
perihelion shift	$\lvert\Lambda\rvert \leq 3 \cdot 10^{-42}\ \mathrm{m}^{-2}$
light deflection	no effect
gravitational time delay	$\lvert\Lambda\rvert \leq 6 \cdot 10^{-24}\ \mathrm{m}^{-2}$
geodetic precession	$\lvert\Lambda\rvert \leq 10^{-27}\ \mathrm{m}^{-2}$
Pioneer anomaly	$\Lambda = -10^{-37}\ \mathrm{m}^{-2}$

Table: Estimates on Λ from Solar system observations.

- Cosmological constant has no influence on Solar system effects
- If some other Λ is assumed to account for the Pioneer anomaly, then it conflicts with Perihelion shift $\Rightarrow$ A constant Λ cannot be responsible for Pioneer anomaly
- May not describe critical orbits (near separatrix) $\rightarrow$ needs exact calculations

Outline

1 One motivation

2 Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
- Solutions
- The equation of motion
- Analytic solution for point particles
- Application to Pioneer anomaly
- Post–Schwarzschild approximation

3 Further applications
- Geodesic equation in higher dimensions
- Further applications

4 Summary

Outline

1 One motivation

2 Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
- Solutions
- The equation of motion
- Analytic solution for point particles
- Application to Pioneer anomaly
- Post–Schwarzschild approximation

3 Further applications
- Geodesic equation in higher dimensions
- Further applications

4 Summary

Literature

Analytic solution for geodesic equation

- Schwarzschild
  Weierstrass elliptic functions
  Hagihara JJGA 1931
- Kerr
  elliptic functions
  Carter 1968, ..., Chandrasekhar 1982
- Reissner–Nordström
  elliptic functions

Outline

1. One motivation

2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation

3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications

4. Summary

Geodesic equation in Schwarzschild–de Sitter space–time

Geodesic equation

$$0 = \frac{d^2 x^\mu}{ds^2} + \left\{ {}^{\;\mu}_{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} \,, \qquad g(u,u) = \epsilon$$

conservation of energy E and angular momentum L

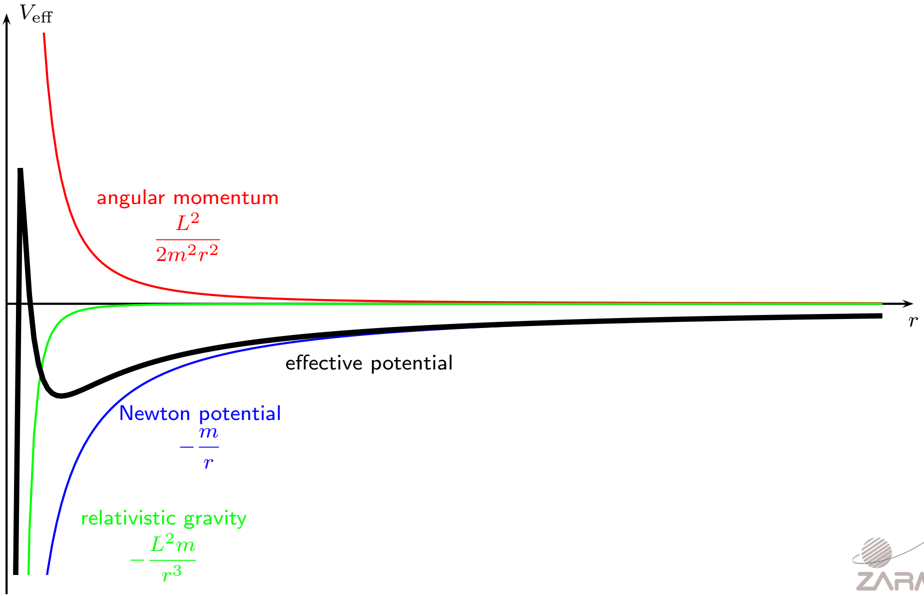
effective equations

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{L^2}\left(E^2 - \left(1 - \frac{r_S}{r} - \frac{1}{3}\Lambda r^2\right)\left(\epsilon + \frac{L^2}{r^2}\right)\right)$$

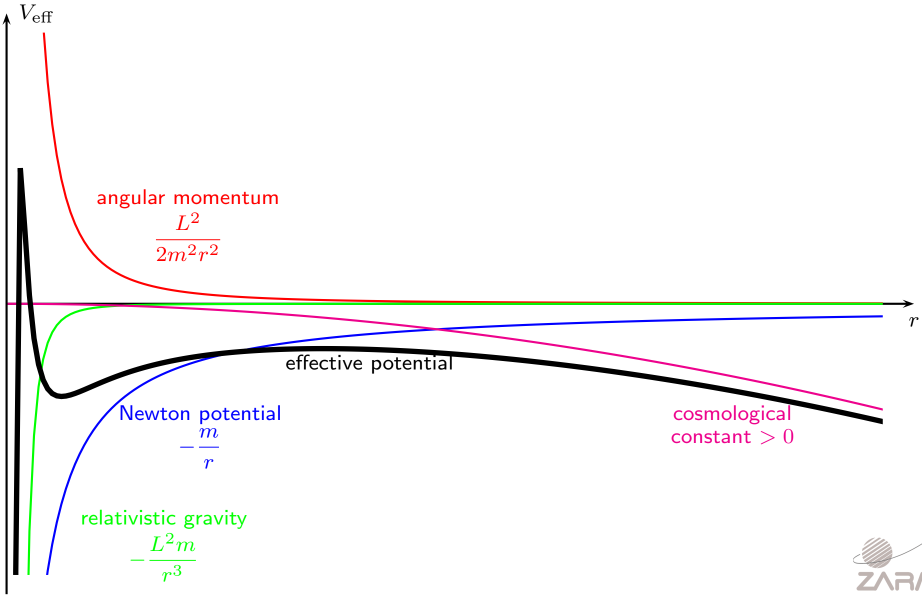
$$\left(\frac{dr}{ds}\right)^2 = E^2 - \left(1 - \frac{r_S}{r} - \frac{1}{3}\Lambda r^2\right)\left(\epsilon + \frac{L^2}{r^2}\right) = E^2 - V_{\text{eff}}(r)$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{1}{E^2}\left(1 - \frac{r_S}{r} - \frac{1}{3}\Lambda r^2\right)^2\left(E^2 - \left(1 - \frac{r_S}{r} - \frac{1}{3}\Lambda r^2\right)\left(\epsilon + \frac{L^2}{r^2}\right)\right)$$

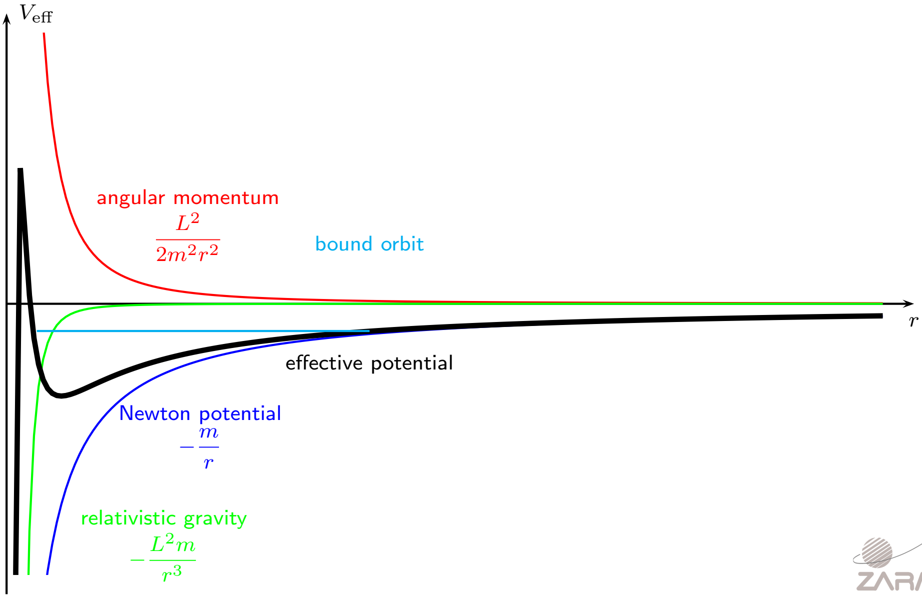
Geodesic equation: effective potential



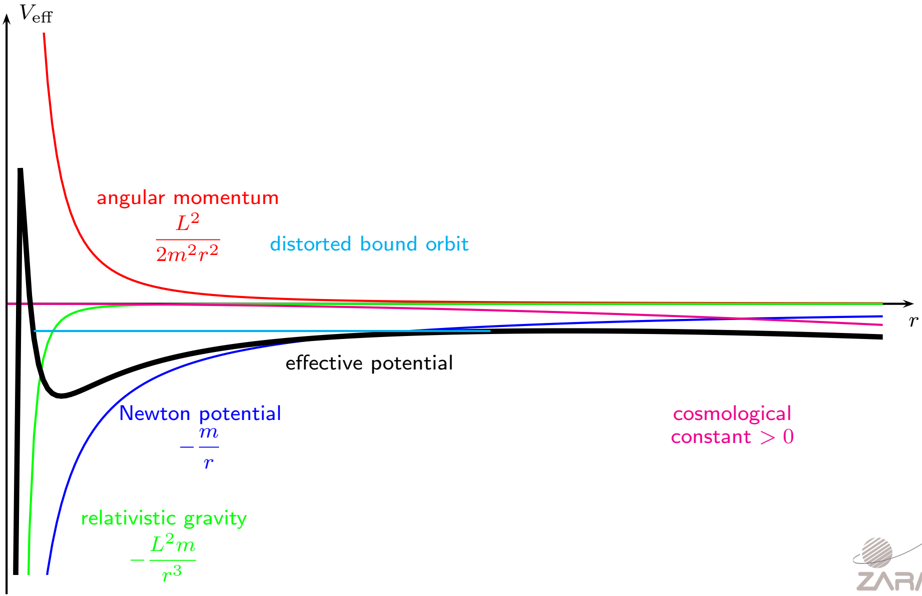
Geodesic equation: effective potential



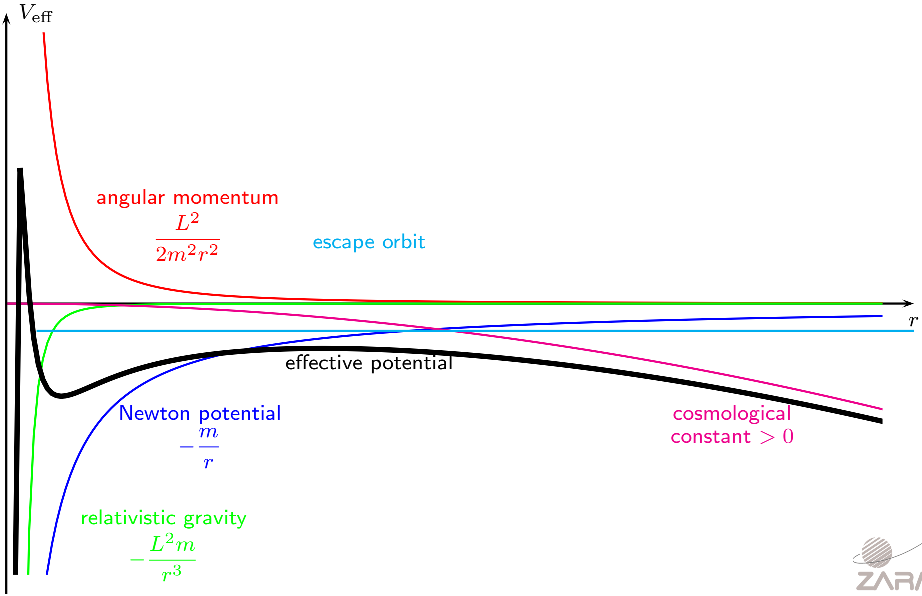
Geodesic equation: effective potential



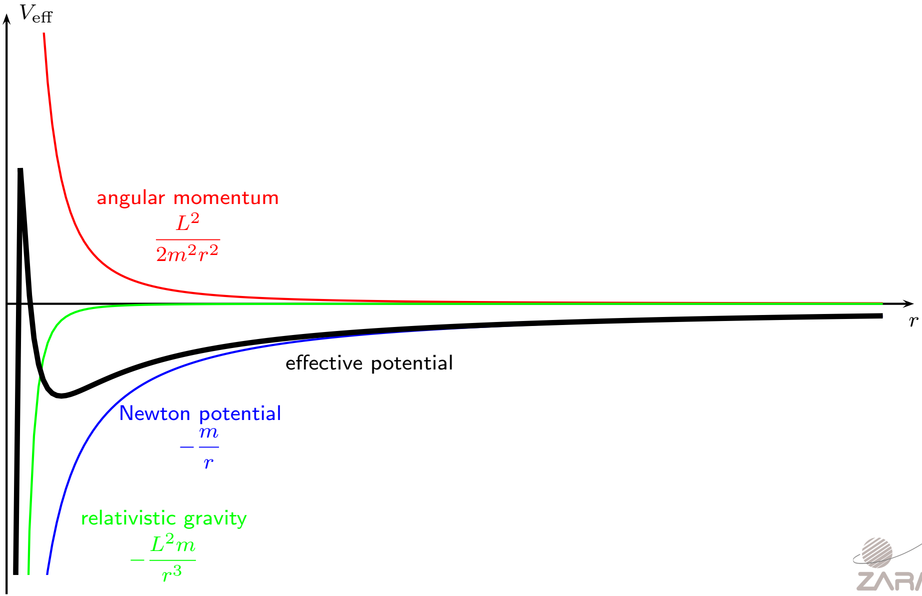
Geodesic equation: effective potential



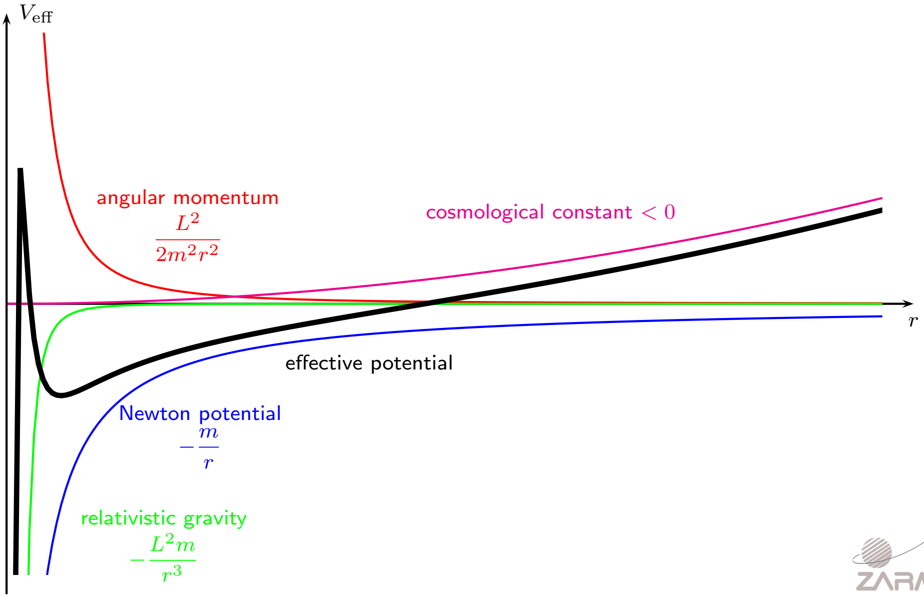
Geodesic equation: effective potential



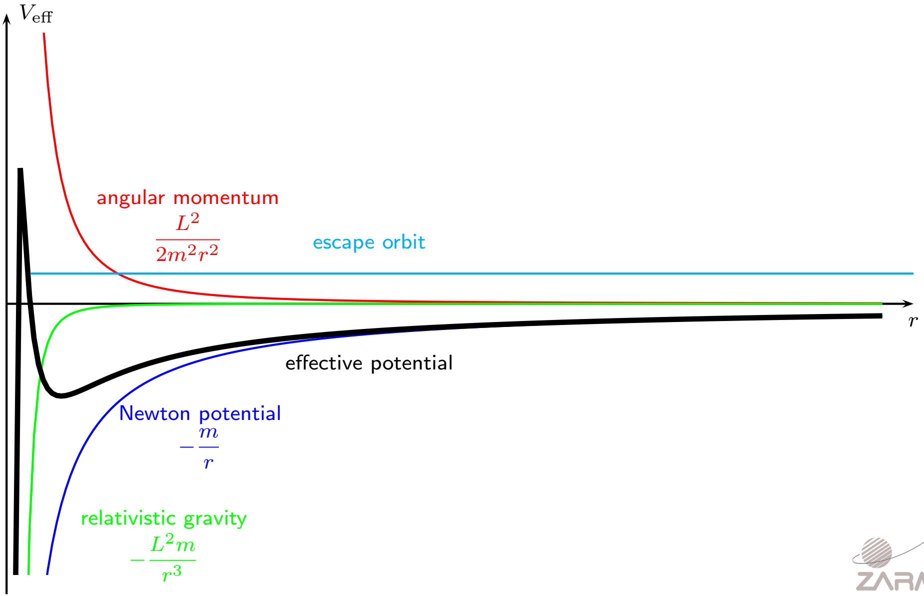
Geodesic equation: effective potential



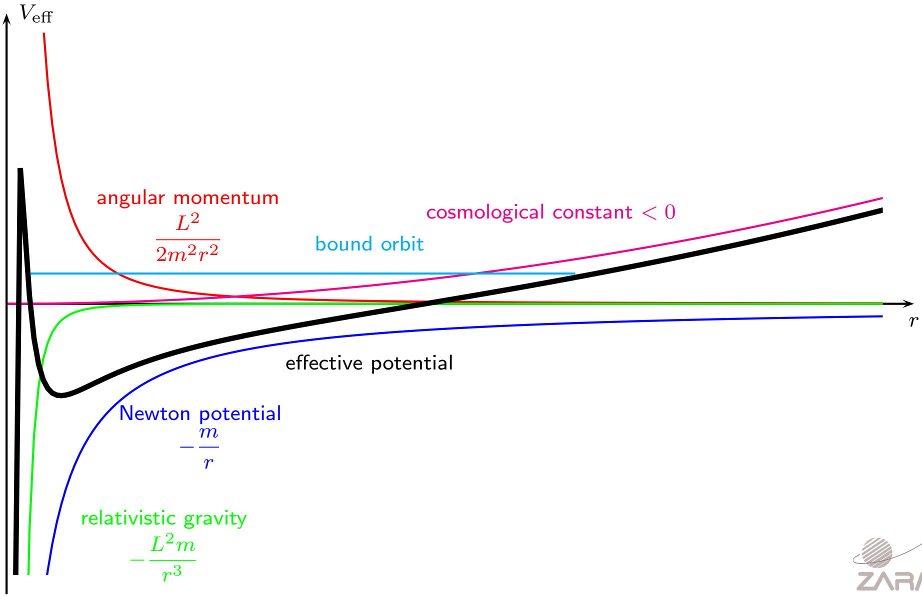
Geodesic equation: effective potential



Geodesic equation: effective potential



Geodesic equation: effective potential



Outline

1. One motivation

2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation

3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications

4. Summary

Geodesic equation in Schwarzschild–de Sitter space–time

Effective equation of motion ($u = r_{\rm S}/r$)

$$\left(u\frac{du}{d\varphi}\right)^2 = u^5 - u^4 + \epsilon\lambda u^3 + (\lambda(\mu-\epsilon)+\rho)\,u^2 + \epsilon\lambda\rho =: P_5(u)$$

with

$$\lambda = \left(\frac{r_{\rm S}}{L}\right)^2, \qquad \mu = E^2, \qquad \rho = \frac{1}{3}\Lambda r_{\rm S}^2$$

- Polynomial of 5th order → beyond elliptic integral: hyperelliptic integral
- P_5 possesses at most 4 real positive zeros
- $\epsilon = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Λ has no influence on light propagation, comp. Ishak & Rindler 2007)
- $\Lambda = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Hagihara JJGA 1931)

Geodesic equation in Schwarzschild–de Sitter space–time

Effective equation of motion $(u = r_S/r)$

$$\left(u\frac{du}{d\varphi}\right)^2 = u^5 - u^4 + \epsilon\lambda u^3 + (\lambda(\mu-\epsilon)+\rho)\,u^2 + \epsilon\lambda\rho =: P_5(u)$$

with

$$\lambda = \left(\frac{r_S}{L}\right)^2, \qquad \mu = E^2, \qquad \rho = \frac{1}{3}\Lambda r_S^2$$

- Polynomial of 5th order → beyond elliptic integral: hyperelliptic integral
- P_5 possesses at most 4 real positive zeros
- $\epsilon = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Λ has no influence on light propagation, comp. Ishak & Rindler 2007)
- $\Lambda = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Hagihara JJGA 1931)

Geodesic equation in Schwarzschild–de Sitter space–time

Effective equation of motion ($u = r_S/r$)

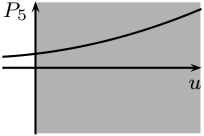
$$\left(u\frac{du}{d\varphi}\right)^2 = u^5 - u^4 + \epsilon\lambda u^3 + (\lambda(\mu - \epsilon) + \rho)\, u^2 + \epsilon\lambda\rho =: P_5(u)$$

with

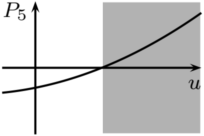
$$\lambda = \left(\frac{r_S}{L}\right)^2 , \qquad \mu = E^2 , \qquad \rho = \frac{1}{3}\Lambda r_S^2$$

- Polynomial of 5th order $\rightarrow$ beyond elliptic integral: hyperelliptic integral
- P_5 possesses at most 4 real positive zeros
- $\epsilon = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Λ has no influence on light propagation, comp. Ishak & Rindler 2007)
- $\Lambda = 0 \Rightarrow P_5(u) = u^2 P_3(u) \Rightarrow$ elliptic function $\wp$ (Hagihara JJGA 1931)

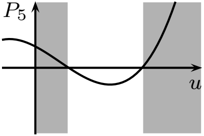
Orbits in Schwarzschild–de Sitter space–time



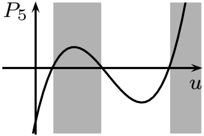
(a) No real positive zero



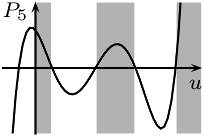
(b) One real positive zero



(c) Two real positive zeros



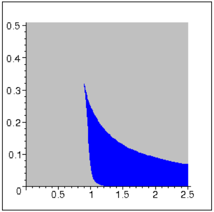
(d) Three real positive zeros



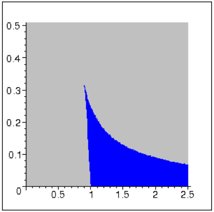
(e) Four real positive zeros

- Five possibilities of having real positive zeros of $P_5(u)$
- Zeros correspond to the positions for which $V_{\text{eff}} = E$
- Bound non–terminating, quasi–periodic orbits (planetary orbits) exist only for three or more positive zeros

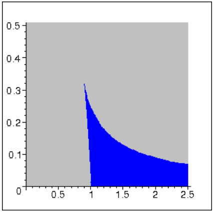
Orbits in Schwarzschild–de Sitter space–time



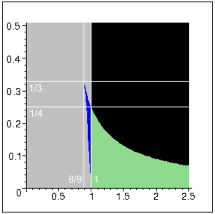
(f) $\Lambda = -10^{-5}$km^{-2}



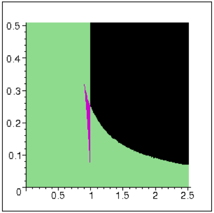
(g) $\Lambda = -10^{-10}$km^{-2}



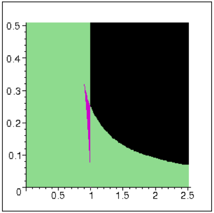
(h) $\Lambda = -10^{-45}$km^{-2}



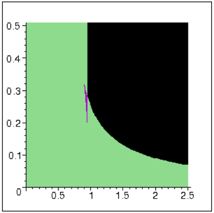
(i) $\Lambda = 0$



(j) $\Lambda = 10^{-45}$km^{-2}



(k) $\Lambda = 10^{-10}$km^{-2}



(l) $\Lambda = 10^{-5}$km^{-2}

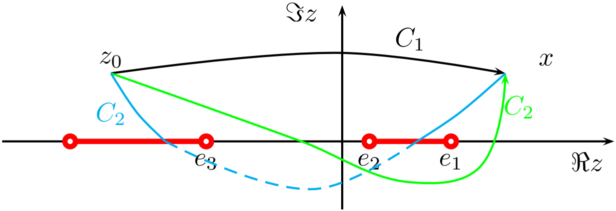
violet = 4, blue = 3, green = 2, gray = 1, black = 0 zeros of P_5.

Analytic solution of geodesic equation in SdS–space–time

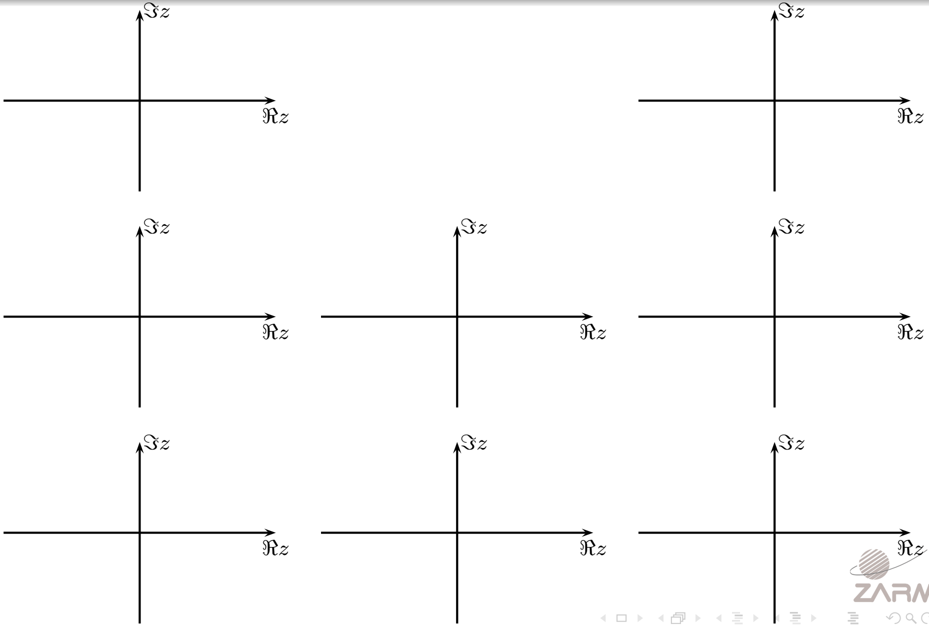
Separation of variables

$$\varphi - \varphi_0 = \int_{u_0}^{u} \frac{u' du'}{\sqrt{P_5(u')}}$$

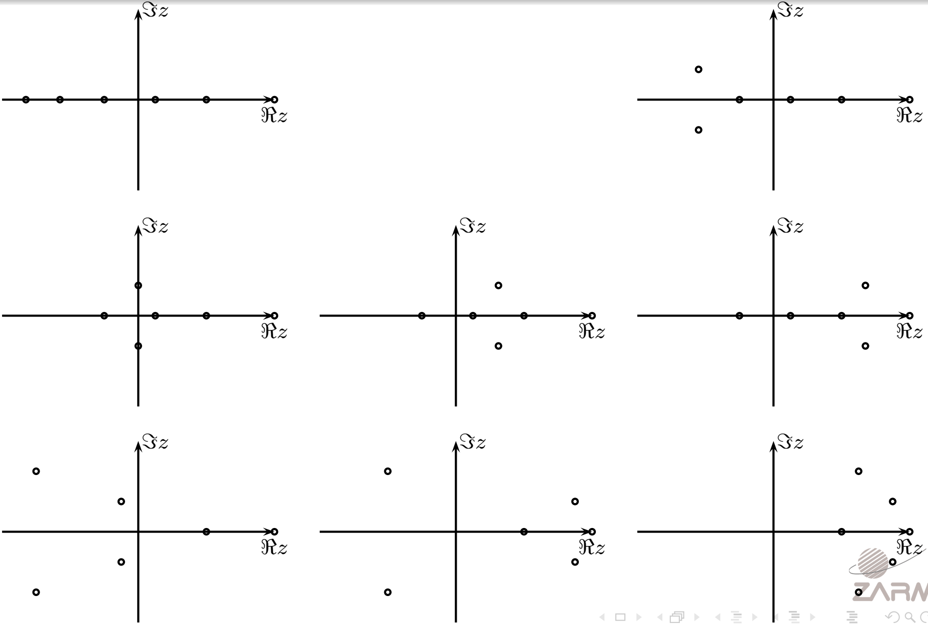
- Not well defined in complex plane
- Looked for: $u = u(\varphi) \leftrightarrow$ inversion problem
- Uniqueness of integration: u is function with 4 periods



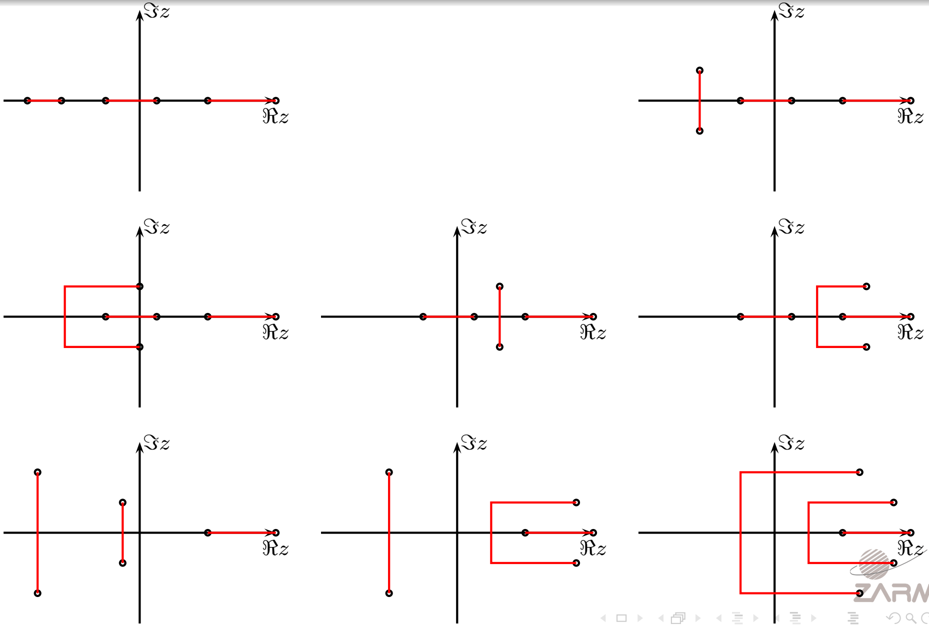
Analytic solution of geodesic equation in SdS–space–time



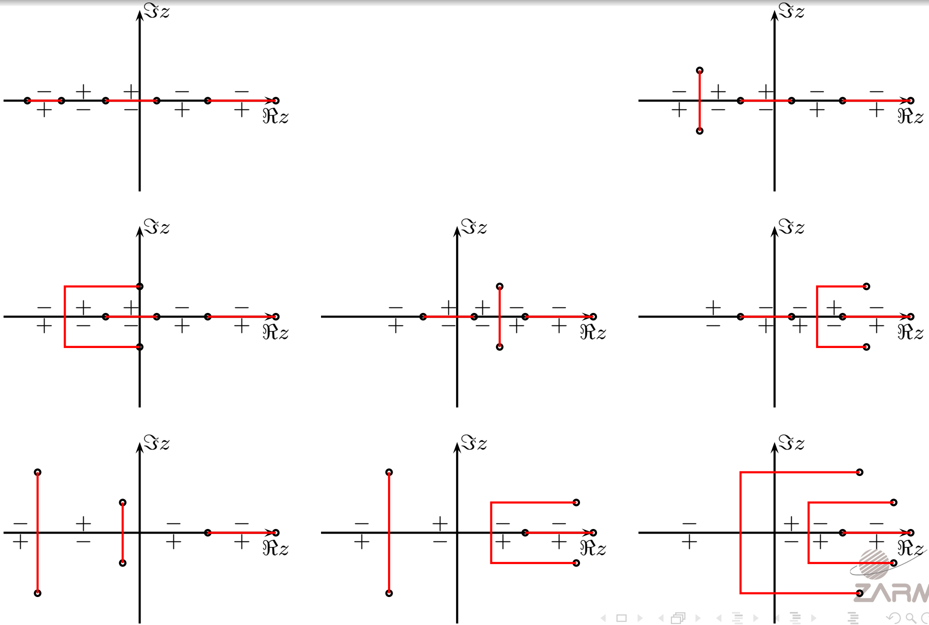
Analytic solution of geodesic equation in SdS–space–time



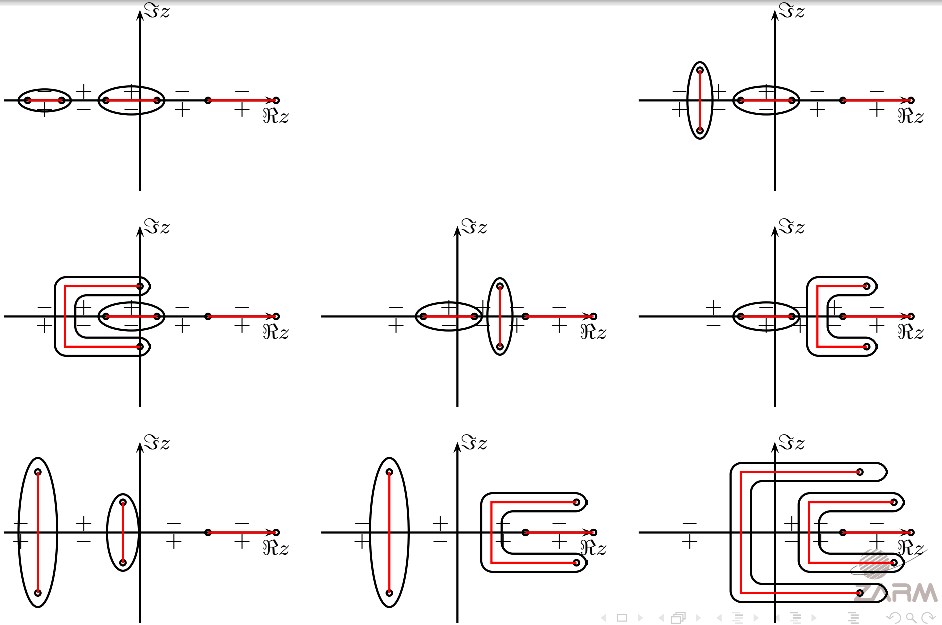
Analytic solution of geodesic equation in SdS–space–time



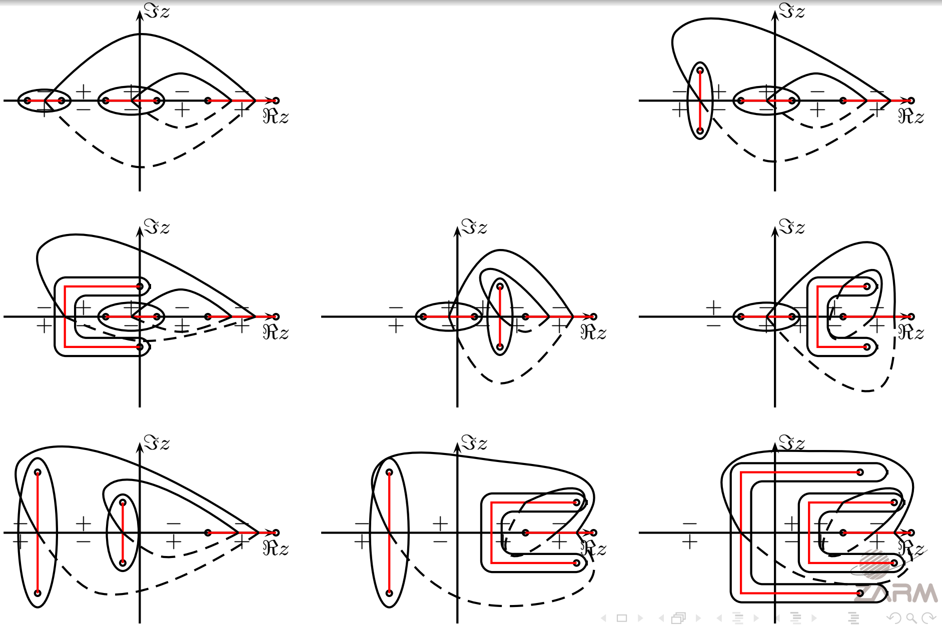
Analytic solution of geodesic equation in SdS–space–time



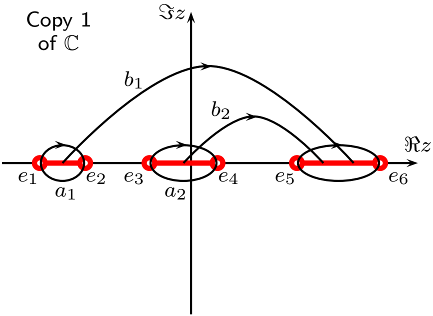
Analytic solution of geodesic equation in SdS–space–time

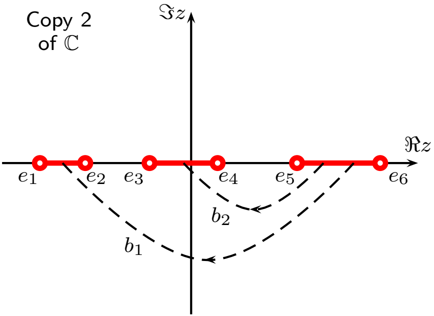


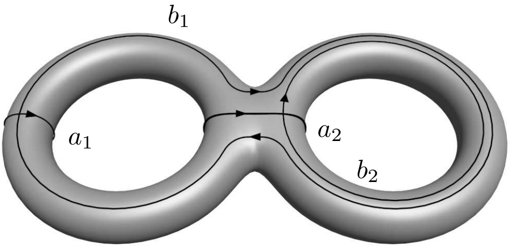
Analytic solution of geodesic equation in SdS–space–time



Analytic solution of geodesic equation in SdS–space–time







$P_5 \leftrightarrow X =$ pretzel

Analytic solution of geodesic equation in SdS–space–time

Holomorphic and associated meromorphic differentials

$$dz_1 := \frac{dx}{\sqrt{P_5(x)}}\,, \qquad dz_2 := \frac{x\,dx}{\sqrt{P_5(x)}}$$

$$dr_1 := \frac{3x^3 - 2x^2 + \lambda x}{4\sqrt{P_5(x)}}dx\,, \qquad dr_2 := \frac{x^2 dx}{4\sqrt{P_5(x)}}$$

Period matrices $(2\omega, 2\omega')$ and $(2\eta, 2\eta')$

$$2\omega_{ij} := \oint_{a_j} dz_i\,, \qquad 2\omega'_{ij} := \oint_{b_j} dz_i$$

$$2\eta_{ij} := -\oint_{a_j} dr_i\,, \qquad 2\eta'_{ij} := -\oint_{b_j} dr_i$$

Normalized differentials and their period matrix

$$d\vec{z} \to d\vec{v} = (2\omega)^{-1} d\vec{z}\,, \qquad (2\omega, 2\omega') \to (1_2, \tau) \quad \text{with} \quad \tau = \omega^{-1}\omega'$$

Analytic solution of geodesic equation in SdS–space–time

Preliminaries – definitions

- Theta function $\vartheta : C^2 \to C$ (for construction of functions with 4 periods)

$$\vartheta(\vec{z};\tau) := \sum_{\vec{m}\in\mathbb{Z}^2} e^{i\pi\vec{m}^t(\tau\vec{m}+2\vec{z})}$$

  - Periodicity: $\vartheta(\vec{z}+1_2\vec{n};\tau) = \vartheta(\vec{z};\tau)$
  - Quasi–periodicity: $\vartheta(\vec{z}+\tau\vec{n};\tau) = e^{-i\pi\vec{n}^t(\tau\vec{n}+2\vec{z})}\vartheta(\vec{z};\tau)$

- Theta function with characteristics $\vec{g},\vec{h} \in \frac{1}{2}\mathbb{Z}^2$

$$\vartheta[\vec{g},\vec{h}](\vec{z};\tau) := e^{i\pi\vec{g}^t(\tau\vec{g}+2\vec{z}+2\vec{h})}\vartheta(\vec{z}+\tau\vec{g}+\vec{h};\tau)$$

- Sigma function $\sigma(\vec{z}) = Ce^{-\frac{1}{2}\vec{z}^t\eta\omega^{-1}\vec{z}}\vartheta\left[\begin{pmatrix}1/2\\1/2\end{pmatrix},\begin{pmatrix}0\\1/2\end{pmatrix}\right]((2\omega)^{-1}\vec{z};\tau)$

- Generalized Weierstrass functions

$$\wp_{ij}(\vec{z}) = -\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\log\sigma(\vec{z}) = \frac{\partial_i\sigma(\vec{z})\partial_j\sigma(\vec{z}) - \sigma(\vec{z})\partial_i\partial_j\sigma(\vec{z})}{\sigma^2(\vec{z})}$$

Analytic solution of geodesic equation in SdS–space–time

Jacobi's inversion problem

Determine $\vec{u}$ for given $\vec{\varphi}$ from $\vec{\varphi} = \vec{A}_{u_0}(\vec{u})$ (Abel map), i.e.

$$\begin{aligned} \varphi_1 &= \int_{u_0}^{u_1} \frac{du}{\sqrt{P_5(u)}} + \int_{u_0}^{u_2} \frac{du}{\sqrt{P_5(u)}} \\ \varphi_2 &= \int_{u_0}^{u_1} \frac{u du}{\sqrt{P_5(u)}} + \int_{u_0}^{u_2} \frac{u du}{\sqrt{P_5(u)}} \end{aligned}$$

Solution of inversion problem

Solution $\vec{u}$ of inversion problem given by

$$\begin{aligned} u_1 + u_2 &= 4\wp_{22}(\vec{\varphi}) \\ u_1\, u_2 &= -4\wp_{12}(\vec{\varphi}) \end{aligned}$$

Our problem: two positions $\vec{u}$, two angles $\vec{\varphi}$ $\quad \rightarrow$ requires reduction

Analytic solution of geodesic equation in SdS–space–time

Jacobi's inversion problem

Determine $\vec{u}$ for given $\vec{\varphi}$ from $\vec{\varphi} = \vec{A}_{u_0}(\vec{u})$ (Abel map), i.e.

$$
\begin{aligned}
\varphi_1 &= \int_{u_0}^{u_1} \frac{du}{\sqrt{P_5(u)}} + \int_{u_0}^{u_2} \frac{du}{\sqrt{P_5(u)}} \\
\varphi_2 &= \int_{u_0}^{u_1} \frac{u\,du}{\sqrt{P_5(u)}} + \int_{u_0}^{u_2} \frac{u\,du}{\sqrt{P_5(u)}}
\end{aligned}
$$

Solution of inversion problem

Solution $\vec{u}$ of inversion problem given by

$$
\begin{aligned}
u_1 + u_2 &= 4\wp_{22}(\vec{\varphi}) \\
u_1\, u_2 &= -4\wp_{12}(\vec{\varphi})
\end{aligned}
$$

Our problem: two positions $\vec{u}$, two angles $\vec{\varphi}$ $\quad\rightarrow$ requires reduction

Analytic solution of geodesic equation in SdS–space–time

Rewrite inversion problem in the form (based on Enolskii & Richter JNS 2005)

$$\vec{\phi} = A_\infty(\vec{u}) \qquad \text{with} \qquad \vec{\phi} = \vec{\varphi} - \int_{u_0}^{\infty} d\vec{z}$$

Extraction of one component of $\vec{u}$ (namely u_2 in Jacobi inv. prob.) through a limit

$$u_1 = \lim_{u_2 \to \infty} \frac{u_1 u_2}{u_1 + u_2} = \frac{\sigma(\vec{\phi}_\infty)\partial_1\partial_2\sigma(\vec{\phi}_\infty) - \partial_1\sigma(\vec{\phi}_\infty)\partial_2\sigma(\vec{\phi}_\infty)}{(\partial_2\sigma)^2(\vec{\phi}_\infty) - \sigma(\vec{\phi}_\infty)\partial_2\partial_2\sigma(\vec{\phi}_\infty)}$$

with

$$\vec{\phi}_\infty = \lim_{u_2 \to \infty} \vec{\phi} = \vec{A}_\infty(\vec{u}_\infty)\,, \qquad \vec{u}_\infty = \begin{pmatrix} u_1 \\ \infty \end{pmatrix}$$

One u but still two φ: Riemann vanishing theorem Mumford 1983:
$(2\omega)^{-1}\vec{A}_\infty(\vec{u}_\infty) \in$ Theta–divisor $\Theta_{K_\infty} = \{\vec{z} \mid \vartheta(\vec{z} + \vec{K}_\infty) = 0\}$ with the vector of Riemann constants $\vec{K}_\infty = \tau \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$

$\Rightarrow \sigma(\vec{\phi}_\infty) = 0$, gives a relation between $\phi_{\infty,1}$ and $\phi_{\infty,2}$.

Analytic solution of geodesic equation in SdS–space–time

Define

$$\vec{\varphi}_\Theta := \begin{pmatrix} x \\ \varphi - \varphi_0' \end{pmatrix} \qquad \text{with} \qquad \varphi_0' = \varphi_0 + \int_{u_0}^{\infty} dz_2$$

and choose x so that $(2\omega)^{-1}\vec{\varphi}_\Theta \in \Theta_{K_\infty}$.

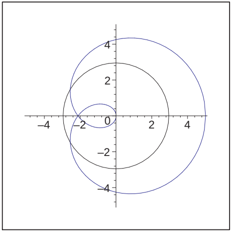
Complete analytic solution of equation of motion in Schwarzschild–de Sitter space–time

$$u(\varphi) = u_1 = -\frac{\partial_1 \sigma(\vec{\varphi}_\Theta)}{\partial_2 \sigma(\vec{\varphi}_\Theta)} \qquad \Rightarrow \qquad \boxed{r(\varphi) = \frac{r_S}{u(\varphi)} = -r_S \frac{\partial_2 \sigma(\vec{\varphi}_\Theta)}{\partial_1 \sigma(\vec{\varphi}_\Theta)}}$$

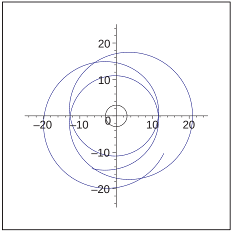
New result

Hackmann & C.L. *Phys. Rev. Lett.* **100** 171101 (2008)
Phys. Rev. **D 78** (2008) in press

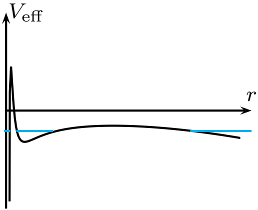
Orbits



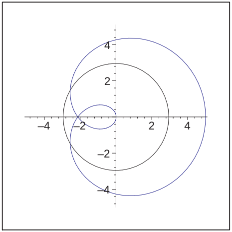
(m) $\Lambda = 0$



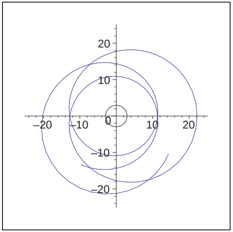
(n) $\Lambda = 0$



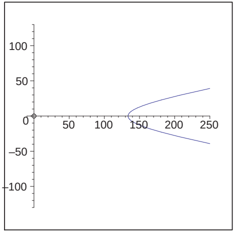
(o) effective potential



(p) $\Lambda > 0$

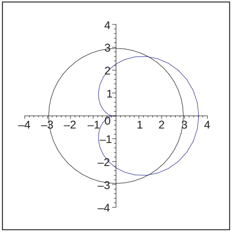


(q) $\Lambda > 0$

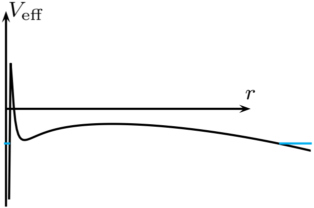


(r) $\Lambda > 0$

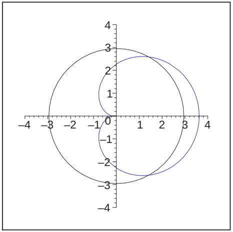
Orbits



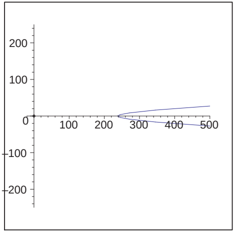
(s) $\Lambda = 0$



(t) effective potential

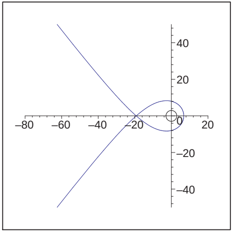


(u) $\Lambda > 0$

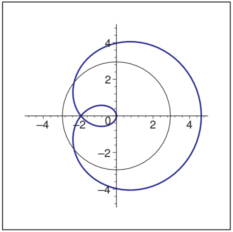


(v) $\Lambda > 0$

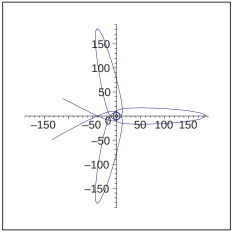
Orbits



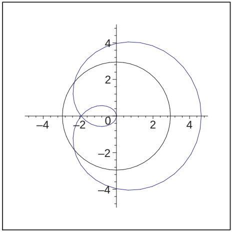
(w) $\Lambda = 0$



(x) $\Lambda = 0$



(y) $\Lambda < 0$



(z) $\Lambda < 0$

Outline

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. Summary

Application to Pioneer anomaly

Influence of Λ on Pioneer satellites (orbital parameters from Nieto & Anderson 2005)

Pioneer 10: $\mu = 1.000\,000\,001\,43\,, \quad \lambda = 2.855\,572\,373\,82 \cdot 10^{-9}$
Pioneer 11: $\mu = 1.000\,000\,001\,22\,, \quad \lambda = 1.340\,740\,574\,59 \cdot 10^{-9}$

- For given angle φ: $r_\Lambda(\varphi) - r_{\Lambda=0}(\varphi) \approx 10^{-3}$ m
- For given distance $r = 65$ AU: $r\varphi_\Lambda(r) - r\varphi_{\Lambda=0}(r) \approx 10^{-5}$ m

$\Rightarrow$ Form of the Pioneer orbits practically does not change.

Cosmological constant cannot be origin of Pioneer anomaly (Hackmann & C.L. *Phys. Rev.* **D 78** (2008) in press).
In principle we need also $\varphi = \varphi(t)$ and Doppler tracking.

Outline

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. Summary

Post–Schwarzschild of perihelion shift

Perihelion shift in Schwarzschild–(anti-)de Sitter (for bound orbit)

$$\delta\varphi_{\text{perihelion}} = 2\pi - \omega_{22} = 2\pi - \oint \frac{x dx}{\sqrt{P_5(x)}}$$

Approximation to first order in Λ

$$\frac{x}{\sqrt{p_5(x)}} = \frac{1}{\sqrt{P_3(x)}} - \frac{2}{3}m^2 \frac{x^2+\lambda}{x^2 P_3(x)\sqrt{P_3(x)}}\Lambda + \mathcal{O}(\Lambda^2)$$

with $P_3(x) = x^3 - x^2 + \lambda x + \lambda(\mu - 1)$ Schwarzschild polynomial
This has to be integrated: gymnastics in elliptic integration

Post–Schwarzschild of perihelion shift

structure of result

$$\oint_{a_2}^{x} \frac{x dx}{\sqrt{P_5(x)}} = \oint_{a_2} \frac{1}{\sqrt{P_3(x)}} - \frac{2}{3} m^2 \oint_{a_2} \frac{x^2 + \lambda}{x^2 P_3(x) \sqrt{P_3(x)}} \Lambda + \mathcal{O}(\Lambda^2)$$

$$= \omega_1 + \Lambda \frac{m^2}{96} \left(\sum_{j=1}^{3} \frac{\eta_1 + \omega_1 z_j}{(\wp''(\rho_j))^2} \left(1 + \frac{\lambda}{\left(4 z_j + \frac{1}{3}\right)^2} \right) + \lambda \left(\frac{2\eta_1 - \frac{1}{6}\omega_1}{16(\wp'(u_0))^2} + \frac{6}{16} \frac{\wp''(u_0)}{(\wp'(u_0))^5} \left(\zeta(u_0) - \eta_1 u_0 \right) \right) \right) + \mathcal{O}(\Lambda^2)$$

- Needs introduction of $r_{\min}$ and $r_{\max}$ or a and e for interpretation
- Relativistic approximation

Outline

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. Summary

Outline

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. Summary

Further space–times: Higher dimensions

(Hackmann, Kagramanova, Kunz & C.L., in preparation)
Formalism can be further applied to some Reissner–Nordström–(anti-)de Sitter space–times in higher dimensions

$$ds^2 = \alpha dt^2 - \alpha^{-1} dr^2 - r^2 d\Omega_{d-2}^2$$

with

$$\alpha = 1 - \frac{m^{d-3}}{r^{d-3}} + \frac{q^{2(d-3)}}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)} r^2$$

Equation of motion

$$\begin{aligned} \left(\frac{dr}{d\varphi}\right)^2 &= \frac{r^4}{L^2} \frac{1}{g_{rr} g_{tt}} \left(E^2 - g_{tt}\left(\epsilon + \frac{L^2}{r^2}\right)\right) \\ &= \frac{r^4}{L^2} \left(E^2 - \left(1 - \frac{m^{d-3}}{r^{d-3}} + \frac{q^{2(d-3)}}{r^{2(d-3)}} - \frac{2\Lambda}{(d-1)(d-2)} r^2\right)\left(\epsilon + \frac{L^2}{r^2}\right)\right) \end{aligned}$$

Substitution $u = \frac{m}{r}$, systematic discussion and study of all subcases $\longrightarrow$ ...

Geodesic equation in higher dimensions

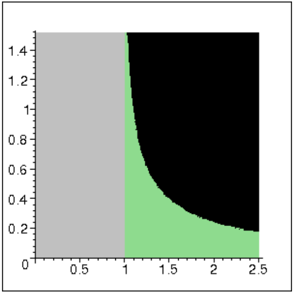
For particles and for light rays

d/ST	S	S(a)dS	RN	RN(a)dS
4	X	X	X	X
5	X	X	X	X
6	X	X	-	-
7	X	X	X	X
8	-	-	-	-
9	X	X	-	-
10	-	-	-	-
11	X	X	-	-
12	-	-	-	-
⋮	⋮	⋮	⋮	⋮

6D–Schwarzschild

Geodesic equation in 6D–Schwarzschild space–time

$$\left(\frac{du}{d\varphi}\right)^2 = u^5 + (\lambda - 1)u^2 + \lambda(\mu - 1) = P_5(u)\,,$$



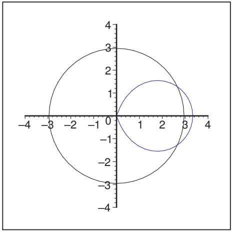
Zeros for 6D–Schwarzschild polynomial

- No bound orbits
- Can be read off from effective potential

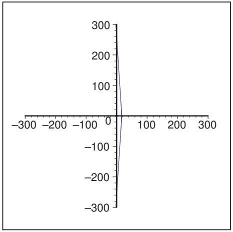
Questions

- In which of these hd space–times do stable orbits exist?
- Deflection angles (Connel & Frolov gr-qc 2008)
- Perihelion shifts

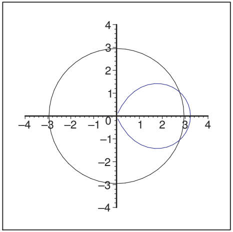
6D–Schwarzschild



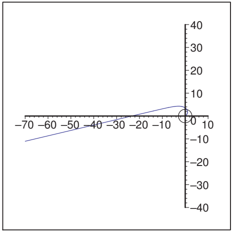
() Terminating orbit $\mu = 1.1$, $\lambda = 0.3$ (green)



() Escape orbit $\mu = 1.1$, $\lambda = 0.3$ (green)



() Terminating orbit $\mu = 0.9$, $\lambda = 0.3$ (gray)



() Infinite terminating orbit $\mu = 1.5$, $\lambda = 0.6$ (black)

Outline

1 One motivation

2 Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
- Solutions
- The equation of motion
- Analytic solution for point particles
- Application to Pioneer anomaly
- Post–Schwarzschild approximation

3 Further applications
- Geodesic equation in higher dimensions
- Further applications

4 Summary

Further applications

- Geodesic equation in Plebanski–Demianski space–time
  - particles: without acceleration
  - light: with acceleration
- Geodesic equations of some modified models of gravity
- Geodesic equations of perturbations of Schwarzschild solution
- Motion of atoms in anharmonic potential in crystals of 6th order $V(x) = \omega^2 x^2 + \omega_4^2 x^4 + \omega_6^2 x^6$
- Theory of spinning top

No scheme known for polynomials of order 7 or higher

Outline

1. One motivation
2. Analytic solution of geodesic equation in Schwarzschild–de Sitter space–time
   - Solutions
   - The equation of motion
   - Analytic solution for point particles
   - Application to Pioneer anomaly
   - Post–Schwarzschild approximation
3. Further applications
   - Geodesic equation in higher dimensions
   - Further applications
4. **Summary**

Summary

Analytic solution for geodesic equation

- Schwarzschild
  Weierstrass elliptic functions
  Hagihara JJGA 1931
- Kerr
  elliptic functions
  Carter 1968, ..., Chandrasekhar 1982
- Reissner–Nordström
  elliptic functions
- Schwarzschild–de Sitter
  hyperelliptic functions
  Hackmann, C.L. PRL 2008, PRD 2008

Part II

Deformation of solids in gravity gradient fields

Outline

5 The problem

6 The model

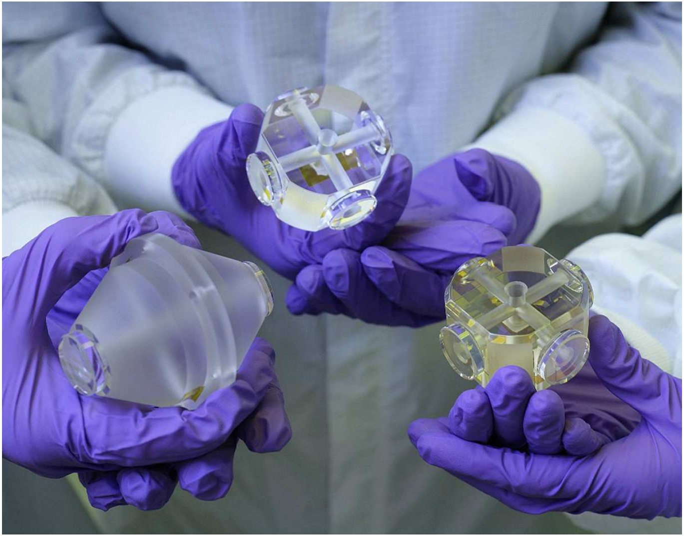
7 Basic equation and boundary conditions

8 Analytical solution

9 Application

10 Summary

Optical resonators



A. Peters

Optical resonators

A. Peters

The problem

Gravity gradient

- Tidal force = difference of acceleration over extension of body

$$\Delta a = \frac{\partial a}{\partial r} L = \frac{\partial^2 U}{\partial r^2} L$$

- Cannot be transformed away, cannot be eliminated
- Has to be calculated and subtracted from signal

Rough estimate of expected effect

- Typically $\Delta a \approx 2 \cdot 10^{-8}\ \mathrm{m/s^2}$ for $L = 5$ cm
- Assumption: Δa acts on top surface of resonator. Hooke's law of elasticity

$$\frac{\Delta L}{L} = \frac{1}{E}\frac{F}{A} = \frac{\rho L}{E}\Delta a \approx 10^{-17}$$

for $F = m\Delta a = \rho L^3 \Delta a$, assuming values for Zerodur.

The problem

In missions like OPTIS, STAR, EGE, ... deformation gives signal of order 10^{-17} .. has to be subtracted

Conclusion: Task

- Real bodies can modeled only numerically
- Numerics must be better than 10^{-19}
- In order to estimate quality of numerical modeling one **needs an analytical solution** for comparison

Outline

5 The problem

6 The model

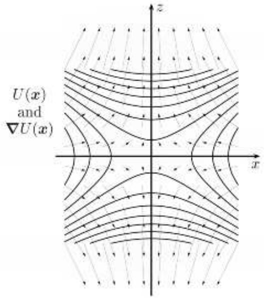
7 Basic equation and boundary conditions

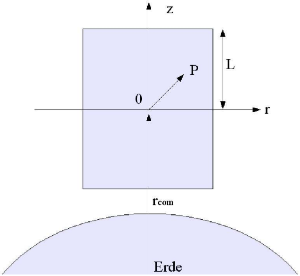
8 Analytical solution

9 Application

10 Summary

The model





- Deforming force in body coordinates (r, z)

$$\boldsymbol{K} = \rho \boldsymbol{\nabla} \left(\frac{GM}{r_{\rm com}} \left(r^2 - 2z^2 \right) \right) ,$$

$G =$ gravitational constant, $M =$ mass of the Earth

Outline

5 The problem

6 The model

7 Basic equation and boundary conditions

8 Analytical solution

9 Application

10 Summary

Basic Equation

- Basic equilibrium equation (K = volume force, gravitaty gradient)

$$D_j \sigma^j{}_i + K_i = 0$$

- Deformation tensor (u = displacement vector field)

$$\epsilon = \frac{1}{2} \mathcal{L}_u g$$

- Hooke's law

$$\sigma^{ij} = c^{ijkl} \epsilon_{kl}$$

Homogenous isotropic elastic solid

$$c^{ijkl} = \mu \left(g^{ik} g^{jl} + g^{jk} g^{il} \right) + \lambda g^{ij} g^{kl}$$

$\Rightarrow$ Lamé-Navier equation ($\Delta = g^{kl} D_k D_l$, $(\text{grad}\,\text{div}\, u)^i = g^{il} D_l D_n u^n$)

$$\boxed{\mu \Delta u^i + (\lambda + \mu)(\text{grad}\,\text{div}\, u)^i + K^i = 0}$$

Homogenous part ($\boldsymbol{K} = 0$) implies the biharmonic equation

$$\Delta\Delta u = 0$$

Boundary Conditions

Typical boundary conditions

- $u^i(x_0) = u_{i0}$, u_{i0} a prescribed initial value
- ${\sigma_i}^j n_j = F_i$, $F =$ surface force (pressure)

Our boundary conditions

- The pressure p is zero and therefore we have the second condition with the surface force $F = 0$.

Outline

5 The problem

6 The model

7 Basic equation and boundary conditions

8 Analytical solution

9 Application

10 Summary

Particular and homogeneous solution

- We search for a solution u as sum of a particular and a homogenous solution: $u = u_p + u_h$
- The particular solution is chosen independently of the boundary conditions
- The boundary conditions with respect to the homogenous equation have to be reformulated.
- The negative boundary values of the particular solution gives the boundary conditions for the homogenous equation: $f^i := \sigma_h^{ij} n_j = -\sigma_p^{ij} n_j$

Particular solution

Ansatz (Lurie 1970)

$$u = \nabla\psi$$

where ψ is a scalar function. This gives a particular solution of the axially symmetric basic equation in the form

$$\begin{aligned} u_r^{\mathrm{p}}(r,z) &= a\left(-\frac{1}{4}r^3 + 2cr\right) \\ u_z^{\mathrm{p}}(r,z) &= a\left(\frac{2}{3}z^3 + 2dz\right) \end{aligned}$$

where $a = \dfrac{1-2\nu}{2\mu(1-2\nu)}\rho\dfrac{GM}{r_{\mathrm{com}}^3}$, c and d arbitrary constants.

Homogeneous boundary conditions

The boundary conditions of the homogenous equation are

$$
\begin{aligned}
f^r(R,z) &= \sigma^{rj} n_j(R,z) = a_2 z^2 + a_0 \\
f^z(R,z) &= \sigma^{zj} n_j(R,z) = 0 \\
f^r(r,\pm L) &= \sigma^{rj} n_j(r,\pm L) = 0 \\
f^z(r,\pm L) &= \sigma^{zj} n_j(r,\pm L) = b_2 r^2 + b_0
\end{aligned}
$$

with some constants a_2, a_0, b_2, b_0. It is not possible to obtain linear boundary conditions.

Homogeneous solution

Love–ansatz in terms of Love function χ

$$
\begin{aligned}
u_r^{\mathrm{h}} &= -\frac{1+\nu}{E}\frac{\partial^2\chi}{\partial r\partial z} \\
u_z^{\mathrm{h}} &= \frac{1+\nu}{E}\left((1-2\nu)\nabla^2\chi + \frac{\partial^2\chi}{\partial r^2} + \frac{1}{r}\frac{\partial\chi}{\partial r}\right)
\end{aligned}
$$

stress tensor

$$
\sigma_{rr}^{\mathrm{h}} = \frac{\partial}{\partial z}\left(\nu\nabla^2\chi - \frac{\partial^2\chi}{\partial r^2}\right) \qquad \sigma_{rz}^{\mathrm{h}} = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right)
$$

$$
\sigma_{zz}^{\mathrm{h}} = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right) \qquad \sigma_{\phi\phi}^{\mathrm{h}} = \frac{\partial}{\partial r}\left(\nu\nabla^2\chi - \frac{1}{r}\frac{\partial\chi}{\partial r}\right) .
$$

The Love function χ necessarily fulfills the biharmonic equation

$$
\nabla^2\nabla^2\chi = 0 .
$$

Homogeneous solution

Ansatz for Love function guided by symmetry of problem (Papkovich-Neuber approach, Meleshko)

$$\chi = B_0 z^3 + \sum_{j=1}^{\infty} \left(A_j \frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j z \frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} \right) \frac{J_0(\lambda_j r)}{\lambda_j^2}$$
$$+ D_0 r^2 z + \sum_{n=1}^{\infty} \left(C_n \frac{I_0(k_n r)}{I_1(k_n R)} + D_n r \frac{I_1(k_n r)}{I_1(k_n R)} \right) \frac{\sin(k_n z)}{k_n^2} .$$

J_0 Bessel functions of first kind and order zero, I_1 are the modified Bessel functions of order one.
$\zeta_j = \lambda_j R$ are zeros of Bessel functions $J_1(\zeta_j) = 0$, and $k_n = \frac{n\pi}{L}$ where n is an integer number.
Ansatz in boundary conditions $\rightarrow$ determination of coefficients

Homogeneous solution

The constants are determined by

$$A_j = -\frac{B_j}{\lambda_j}\left(2\nu + \lambda_j L \coth(\lambda_j L)\right)$$

$$C_n = -\frac{D_n}{k_n}\left((2-2\nu) + k_n R \frac{I_0(k_n R)}{I_1(k_n R)}\right)$$

$$D_n = -\frac{1}{k_n R\left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1\right) - \frac{2-2\nu}{k_n R}}\left(\frac{4\hat{c}_2(-1)^n}{k_n^2} + \sum_{j=1}^{\infty} B_j (-1)^n \frac{4\lambda_j k_n^2}{L(k_n^2+\lambda_j^2)^2} J_0(\lambda_j R)\right)$$

$$B_j = \frac{1}{J_0(\lambda_j R)\left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)}\right)}\left(\frac{2}{\lambda_j^2}\hat{c}_2 - \sum_{n=1}^{\infty}(-1)^n D_n k_n \left(\frac{4\lambda_j^2}{R(k_n^2+\lambda_j^2)^2}\right)\right)$$

$$B_0 = \frac{1}{6}\left(-\hat{c}_1 + \hat{c}_2\left(\frac{R^2}{4} - L^2\right) - S\right) - D_0$$

$$D_0 = \frac{1}{2(1+\nu)}\left((1-\nu)\left(\hat{c}_1 + \hat{c}_2 L^2 + S\right) + \nu\hat{c}_2\frac{R^2}{4} - \mathcal{Z}\right)$$

Homogeneous solution

with

$$\hat{c}_0 = \gamma \left(\frac{4c\nu}{1-\nu} + 2(L^2 + d) \right) \qquad \hat{c}_1 = \gamma \left(\frac{(2\nu - 3)}{4(1-\nu)} R^2 + \frac{2(c + \nu d)}{1-\nu} \right) \qquad \hat{c}_2 = \gamma \frac{2\nu}{1-\nu}$$

and

$$\mathcal{S} = \mathcal{R} + \mathcal{Z}$$

$$\mathcal{R} = \sum_{j=1}^{\infty} B_j \left(\frac{(1 - \lambda_j L \coth(\lambda_j L))}{\sinh(\lambda_j L)} \right) J_0(\lambda_j R) + \sum_{n=1}^{\infty} D_n \left(k_n R \left(\frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1 \right) - \frac{2 - 2\nu}{k_n R} \right)$$

$$\mathcal{Z} = \sum_{j=1}^{\infty} B_j \left(\coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)} \right) + \sum_{n=1}^{\infty} (-1)^n D_n \left(2 - k_n R \frac{I_0(k_n R)}{I_1(k_n R)} \right) \frac{1}{I_1(k_n R)}$$

solution forms infinite system of equations whose convergence can be shown
$\Rightarrow$ analytic solution

First analytical solution for deformation of elastic body under gravity gradient volume force, S. Scheithauer & C.L., *Class. Quantum Grav.* **23**, 7273 (2006)

Can be used for analyzing mirrors in gw interferometers.

Outline

5 The problem

6 The model

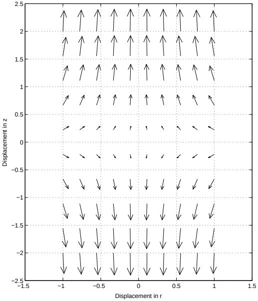
7 Basic equation and boundary conditions

8 Analytical solution

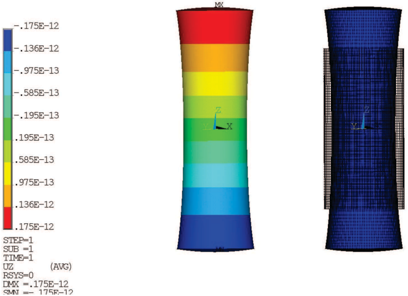
9 Application

10 Summary

Analytical Solution



Displacement field. The infinite sums in the analytical solution were expanded to $N = J = 1700$.



FEM solution. Deformation scaled by a factor of $6 \cdot 10^{13}$. Right: Deformed cylinder shape and original finite element mesh. Left: the scale shows the z displacements.

Analytical Solution

Conclusion

- Comparison between analytical and FEM solution: confirmation
- Supports quality of Pioneer thermal analysis at ZARM using similar FEM methods ....

Outline

5 The problem

6 The model

7 Basic equation and boundary conditions

8 Analytical solution

9 Application

10 Summary

Summary

Two pieces of mathematics which would not have been carried through without motivation from space related problems:

Analytic solutions of

- Geodesic equation in Schwarzschild–de Sitter space–times
- Deformation of elastic bodies in gravity gradient field

The End – Thanks

Thank you for your attention

Thanks to

- H. Dittus
- E. Hackmann, S. Scheithauer
- V. Kagramanova, J. Kunz
- W. Fischer, P. Richter

- German Research Foundation DFG
- German Aerospace Center DLR
- Center of excellence QUEST