Maximally Supersymmetric Gauge Theory at Four Loops

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with Z. Bern, M. Czakon, L. Dixon, & V. Smirnov

QCD

- Nature's gift: a fully consistent physical theory
- Only now, thirty years after the discovery of asymptotic freedom, are we approaching a detailed and explicit understanding of how to do precision theory around zero coupling
- Can compute some static strong-coupling quantities via lattice
- Otherwise, only limited exploration of high-density and hot regimes
- To understand the theory quantitatively in all regimes, we seek additional structure
- String theory returning to its roots

- Study $\mathcal{N} = 4$ large-N gauge theories: maximal supersymmetry as a laboratory for learning about less-symmetric theories

- Connection to string theory via AdS/CFT a model for strong-coupling physics more generally?

- Here, we're seeking hints for additional structure in $\mathcal{N} = 8$ supergravity $\Rightarrow$ Bern, Dixon, and Ita's talks
- Don't need to show that $\mathcal{N}$=8 is finite, it suffices to show additional structure that it be less divergent than expected from pure supersymmetry (superspace) arguments

Descriptions of $\mathcal{N}=4$ SUSY Gauge Theory

- A Feynman path integral

- Boundary CFT of IIB string theory on $AdS_5 \times S^5$

Maldacena (1997)

- Spin-chain model

Minahan & Zarembo (2002)

- Twistor-space topological string B model

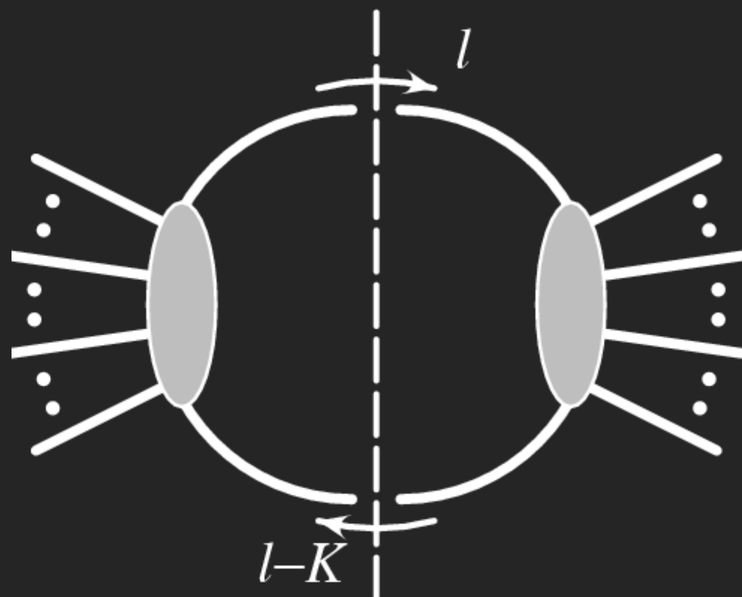
Witten (2003)

Roiban, Spradlin, & Volovich (2004); Berkovits & Motl (2004)

- Comparison between string theory and gauge theory in the limit of rapidly-spinning strings

  Minahan, Zarembo, Staudacher, Beisert, Kristjansen, Eden, … (2003–2006)
- Consider single-trace operators $\mathrm{Tr}(D^s \mathcal{Z}^L)$
- Anomalous dimensions given by Hamiltonian of spin chain
- Solved by Bethe ansatz
- Related to masses of string states ($L \leftrightarrow SO(6)$ charge)
- Large spin limit $s \to \infty$ corresponds to so-called "soft anomalous dimension" accessible in perturbative computation
- Perturbative expansion possible on both sides of correspondence

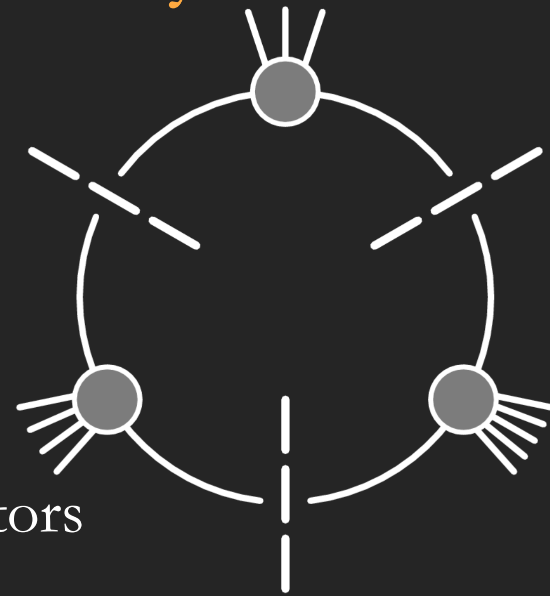
Unitarity-Based Calculations



Bern, Dixon, Dunbar, & DAK (1994)

$$A^{\text{1-loop}} = \sum_{\text{cuts } K^2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} A^{\text{tree}}_{\text{left}} \frac{i}{(\ell-K)^2} A^{\text{tree}}_{\text{right}}$$

Generalized Unitarity



- Can sew together more than two tree amplitudes
- Corresponds to 'leading singularities'

- Isolates contributions of a smaller set of integrals: only integrals with propagators corresponding to cuts will show up

Bern, Dixon, DAK (1997)

- Example: in triple cut, only boxes and triangles will contribute

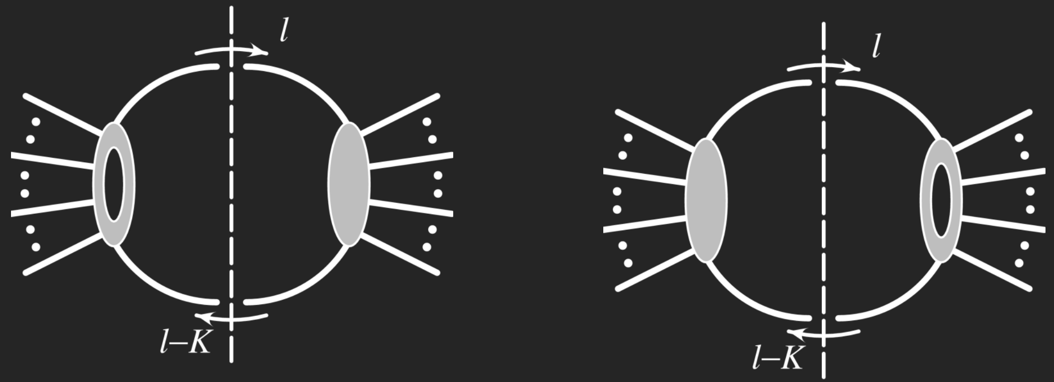
(in $\mathcal{N} = 4$, a smaller set of boxes)

- Combine with use of complex momenta to determine box coeffs directly in terms of tree amplitudes

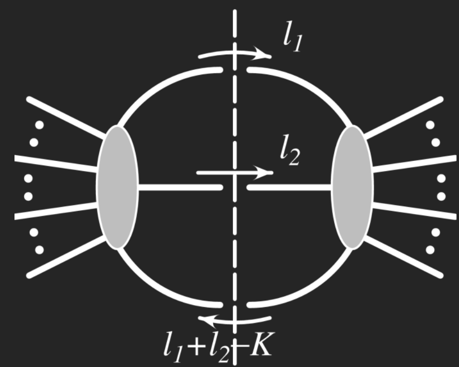
Britto, Cachazo, & Feng (2004)

Unitarity Method at Higher Loops

- Loop amplitudes on either side of the cut



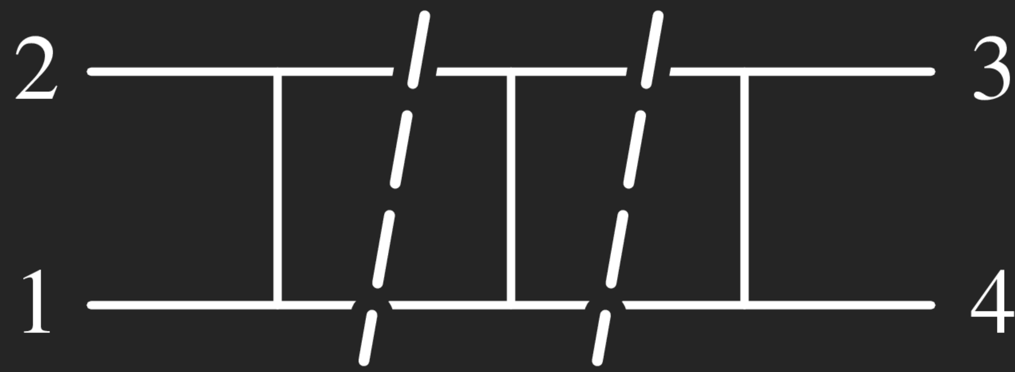
- Multi-particle cuts in addition to two-particle cuts



- Find integrand/integral with given cuts in all channels

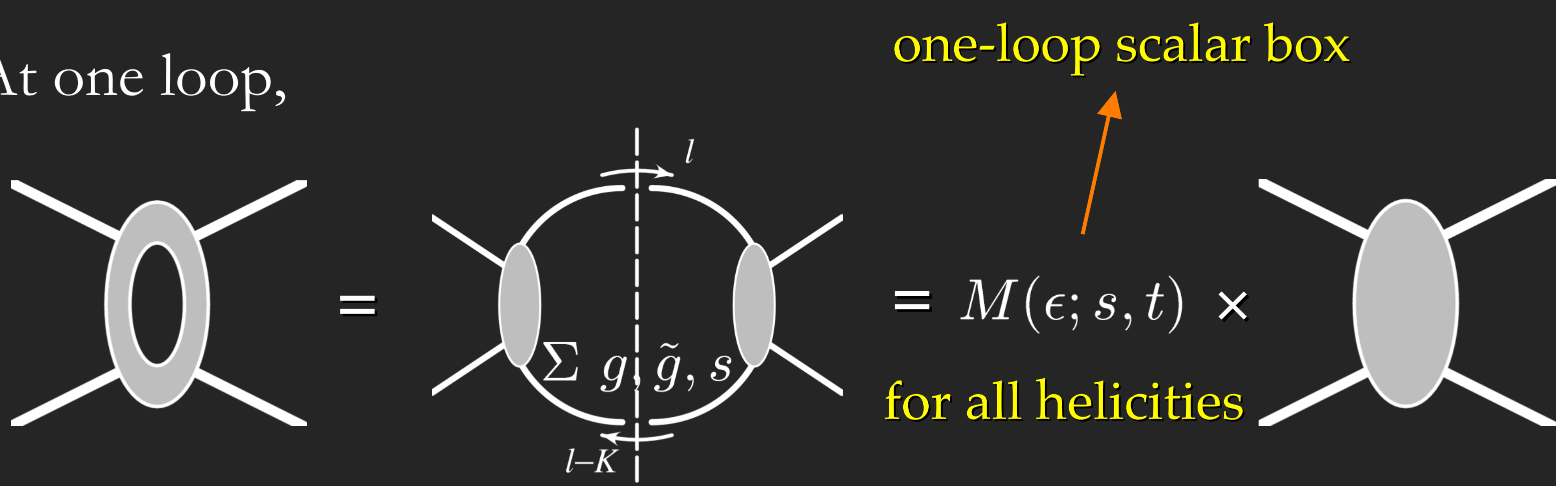
Generalized Cuts

- In practice, replace loop amplitudes by their cuts too



$\mathcal{N}=4$ Cuts at Two Loops

- At one loop,

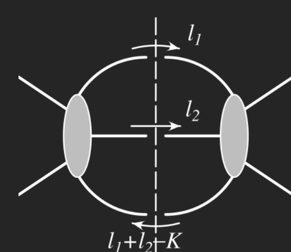


Green, Schwarz, & Brink (1982)

- Two-particle cuts iterate to all orders

Bern, Rozowsky, & Yan (1997)

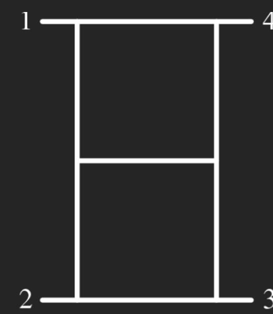
- Three-particle cuts give no new information for the four-point amplitude



Two-Loop Four-Point Result

- Integrand known

$$-st A_4^{\text{tree}} \Big\{ s$$


$$+t$$


$$\Big\}$$

Bern, Rozowsky, & Yan (1997)

- Integrals known by 2000 ⇒ could have just evaluated

- Singular structure is an excellent guide

Sterman & Magnea (1990); Catani (1998); Sterman & Tejeda-Yeomans (2002)

Two-loop Double Box

$$x = \frac{s}{t}$$

$$
\begin{aligned}
&-\frac{4}{\epsilon^4} + \frac{5\ln x}{\epsilon^3} - \frac{1}{\epsilon^2}\left(2\ln^2 x - \frac{5}{2}\pi^2\right) \\
&-\frac{1}{\epsilon}\bigg(4\,\mathrm{Li}_3(-x) - 4\ln x\,\mathrm{Li}_2(-x) - (\ln^2 x + \pi^2)\ln(1+x) \\
&\qquad + \frac{2}{3}\ln^3 x + \frac{11}{2}\pi^2 \ln x - \frac{65}{3}\zeta(3)\bigg) \\
&- 4\big(S_{2,2}(-x) - \ln x S_{1,2}(-x)\big) + 44\,\mathrm{Li}_4(-x) \\
&- 4\big(\ln(1+x) + 6\ln x\big)\,\mathrm{Li}_3(-x) \\
&+ 2\big(\ln^2 x + 2\ln x\ln(1+x) + \frac{10}{3}\pi^2\big)\,\mathrm{Li}_2(-x) \\
&+ \big(\ln^2 x + \pi^2\big)\ln^2(1+x) - \frac{2}{3}(4\ln^3 x + 5\pi^2\ln x - 6\zeta(3))\ln(1+x) \\
&+ \frac{4}{3}\ln^4 x + 6\pi^2\ln^2 x - \frac{88}{3}\zeta(3)\ln x + \frac{29}{30}\pi^4 + \mathcal{O}(\epsilon)
\end{aligned}
$$

Smirnov (1999)

Physics is 90% mental, the other half is hard work — Yogi Berra

Transcendentality

- Also called 'polylog weight'

$$\begin{aligned} \mathrm{Tran}[\ln] &= \mathrm{Tran}[\pi] = 1 \\ \mathrm{Tran}[\mathrm{Li}_n] &= n \end{aligned}$$

$$\mathrm{Tran}\left[\sum_{0<j_1<j_2<\cdots<j_m}^{\infty} \frac{x_i^{j_i}}{j_1^{n_1} j_2^{n_2} \cdots j_m^{n_m}}\right] = n_1 + \cdots + n_m$$

- $\mathcal{N} = 4$ SUSY has maximal transcendentality = 2 × loop order
- QCD has mixed transcendentality: from 0 to maximal

- Singular structure is an excellent guide

Sterman & Magnea (1990); Catani (1998); Sterman & Tejeda-Yeomans (2002)

- Kinoshita–Lee–Nauenberg theorem guarantees cancellations between IR singular parts of virtual contributions and integrals over real emission

- The latter arise from regions where $(n+1)$-point amplitude factorizes into singular emission × n-point amplitude

- Consistency possible only if one-loop virtual amplitude has IR singularities proportional to tree amplitude

- At higher loops, singularities ($1/^{\mathbf{2}}2$ per loop order) are proportional to lower-loop amplitudes

- Leading-most $1/^{\mathbf{2}}2l$ are proportional to the tree, and are related to uncorrelated multiple emission; cancel along with (l-1)-loop × 1-loop, (l-2)-loop × 2-loop, etc.

$$\frac{1}{\ell!}\left[V^{\text{1-loop}}\right]^{\ell} A_n^{\text{tree}}(1,\ldots,n)+\cdots$$

- At two loops, the coefficients of $1/^{\mathbf{2}}2$ and $1/^{\mathbf{2}}$ are "new": contain parts related to correlated double emission, not related to lower order

- At each order, the coefficients of $1/^{\mathbf{2}}2$ and $1/^{\mathbf{2}}$ are "new": anomalous dimensions

- IR-singular terms exponentiate

$$\exp\left[-\frac{1}{8}\sum_{l=1}^{\infty} c^l \alpha_s^l \left(\frac{\gamma^{(l)}}{l^2\epsilon^2} + \frac{2G_0}{l\epsilon}\right) \sum_{\substack{\text{color-connected}\\ \text{pairs}}} \left(\frac{\mu^2}{s_{\text{pair}}}\right)^{l\epsilon}\right] \times \text{Finite}$$

- Soft or "cusp" anomalous dimension: large-spin limit of trace-operator anomalous dimension

Sterman & Magnea (1990)

$$A_4^{(2)} = \left[\frac{1}{2}\Big(I_4^{(1)}(s,t;\epsilon)\Big)^2 + f(\epsilon) I_4^{(1)}(s,t;2\epsilon)\right]\Bigg|_{\text{poles}} \times \text{Finite}$$

$$+\text{Finite} + \mathcal{O}(\epsilon)$$

- True for all gauge theories

Iteration Relation in $\mathcal{N}=4$

- Look at corrections to MHV amplitudes $(- + \cdots - + \cdots)$, at leading order in N_c $(M^L \equiv A^{L-\text{loop}}/A^{\text{tree}})$

$$M_4^{(2)}(s,t;\epsilon) = \frac{1}{2}\Big(M_4^{(1)}(s,t;\epsilon)\Big)^2 + f^{(2)}(\epsilon)M_4^{(1)}(s,t;2\epsilon) - \frac{1}{2}\zeta_2^2 + \mathcal{O}(\epsilon)$$

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2 + \cdots)$$

- Including finite terms

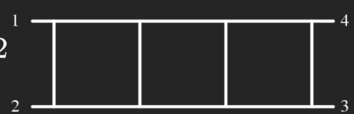
Anastasiou, Bern, Dixon, & DAK (2003)

- Requires non-trivial cancellations not predicted by pure supersymmetry or superspace arguments

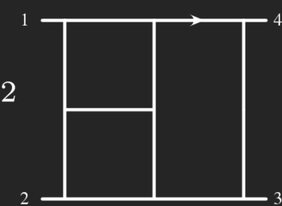
$\mathcal{N} = 4$ Integrand at Higher Loops

Bern, Rozowsky, & Yan (1997)

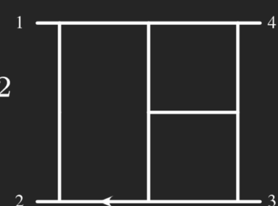
$-istA_4^{\text{tree}} \Big\{ s^2$



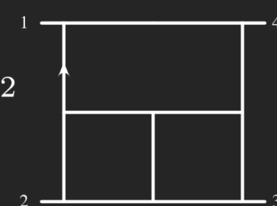
$+\, s(\ell + k_1)^2$



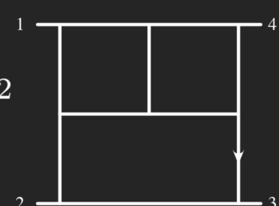
$+\, s(\ell + k_3)^2$



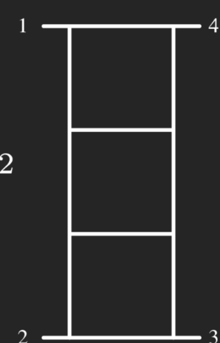
$+\, t(\ell + k_2)^2$



$+\, t(\ell + k_4)^2$



$+\, t^2$



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Iteration Relation

- Confirmed at three loops

Bern, Dixon, & Smirnov (2005)

$$M_4^{(3)}(\epsilon) = -\frac{1}{3}\Big(M_4^{(1)}(\epsilon)\Big)^3 + M_4^{(1)}(\epsilon)M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)\, M_4^{(1)}(3\epsilon) + C^{(3)}$$

$$f^{(3)}(\epsilon) = \frac{11}{4}\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)$$

$$C^{(3)}(\epsilon) = \Big(\frac{341}{216} + \frac{2}{9}c_1\Big)\zeta_6 + \Big(-\frac{17}{9} + \frac{2}{9}c_2\Big)\zeta_3^2$$

highest polylog weight

- Can extract 3-loop anomalous dimension & compare to Kotikov, Lipatov, Onishchenko and Velizhanin “extraction” from Moch, Vermaseren & Vogt result

$$\gamma_{\mathcal{N}=4} = \gamma_{\rm QCD}\Big|_{\rm leading\ \ transcendentality}$$

All-Loop Conjecture

Bern, Dixon, & Smirnov (2005)

$$
\begin{aligned}
M_n &= 1 + \sum_{l=1}^{\infty} c^l \alpha_s^l M_n^{(l)}(\epsilon) \\
&= \exp\left[\sum_{l=1}^{\infty} c^l \alpha_s^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right)\right]
\end{aligned}
$$

$\mathcal{O}(\epsilon)$

Beyond Four-Point Amplitude

- Iteration relation holds for splitting amplitude
- Would extend relation to all-n MHV if it were true for the five-point amplitude

- It is!

Cachazo, Spradlin, & Volovich (2006)

Bern, Czakon, DAK, Roiban, & Smirnov (2006)

$\Rightarrow$ Spradlin's talk

Four Loops

- Does the iteration relation hold at this order?
- Stay tuned… need $O(\epsilon^{0})$

- Computation of $\gamma_{\mathcal{N}=4}$ only needs $O(\epsilon^{-2})$ terms

Calculation

Dick [Feynman]'s method is this. You write down the problem. You think very hard. Then you write down the answer. — Murray Gell-Mann

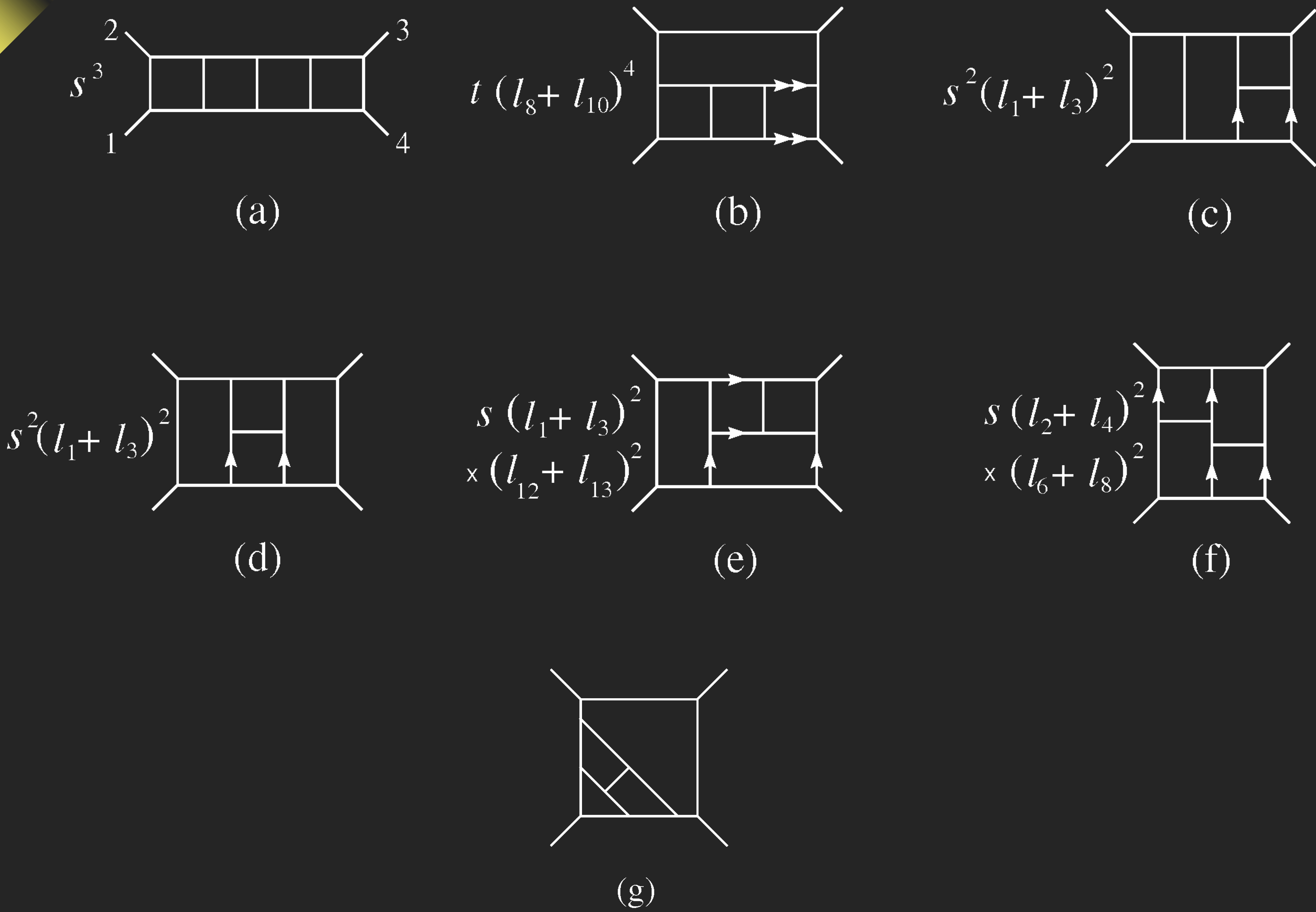
- Integral set
- Unitarity
- Calculating integrals

Integrals

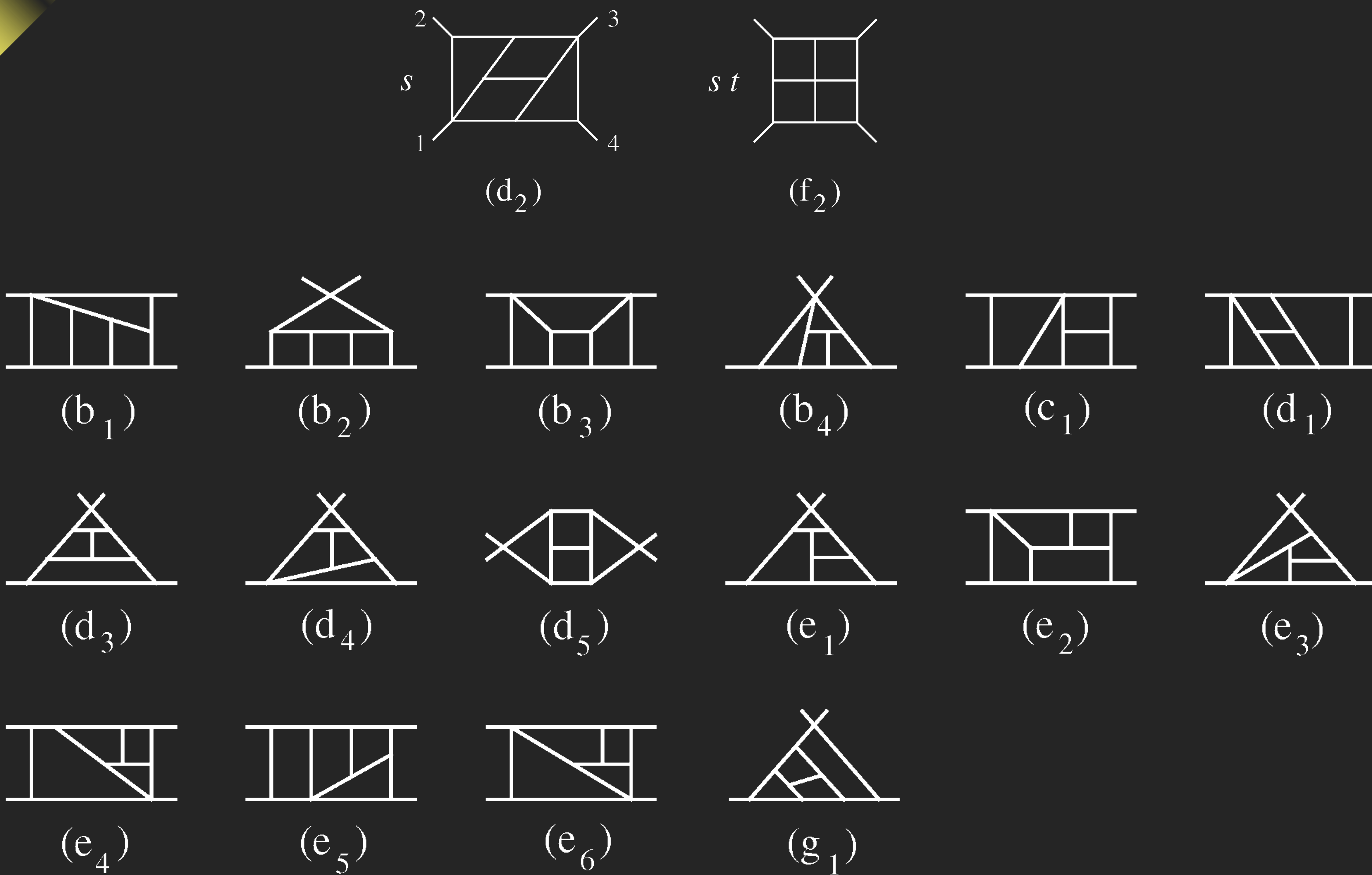
- Start with all four-point four-loop integrals with no bubble or triangle subgraphs (expected to cancel in $\mathcal{N}$=4)

⇒ 7 master topologies (only three-point vertices)

⇒ 25 potential integrals (others obtained by canceling propagators)



s^3
2
3
1
4
(a)
$t\,(l_8 + l_{10})^4$
(b)
$s^2(l_1 + l_3)^2$
(c)
$s^2(l_1 + l_3)^2$
(d)
$s\,(l_1 + l_3)^2$
$\times\,(l_{12} + l_{13})^2$
(e)
$s\,(l_2 + l_4)^2$
$\times\,(l_6 + l_8)^2$
(f)
(g)


2
3
s
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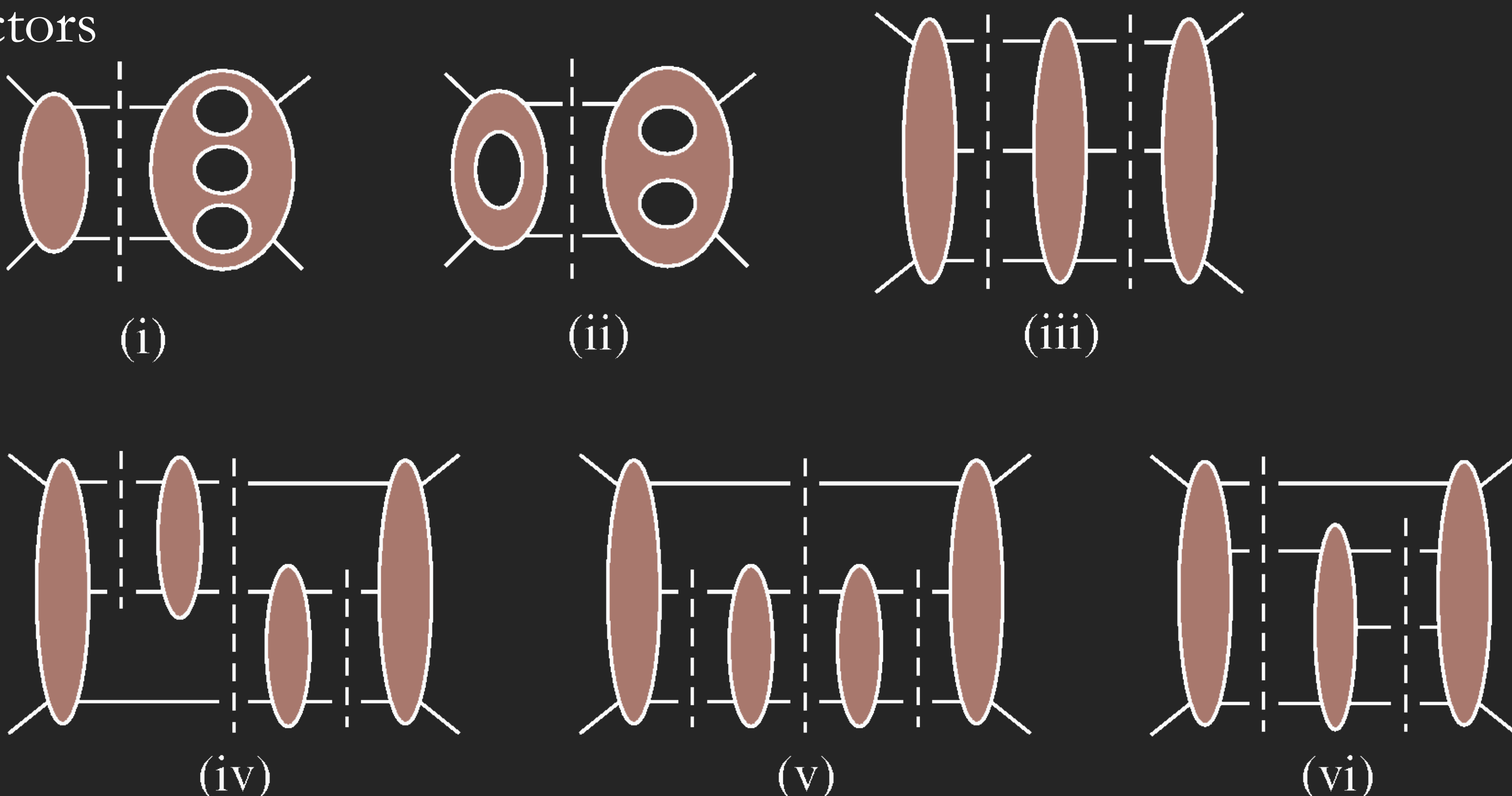
(d2)
s t
(f2)
(b1)
(b2)
(b3)
(b4)
(c1)
(d1)
(d3)
(d4)
(d5)
(e1)
(e2)
(e3)
(e4)
(e5)
(e6)
(g1)

Cuts

- Compute a set of six cuts, including multiple cuts to determine which integrals are actually present, and with which numerator factors

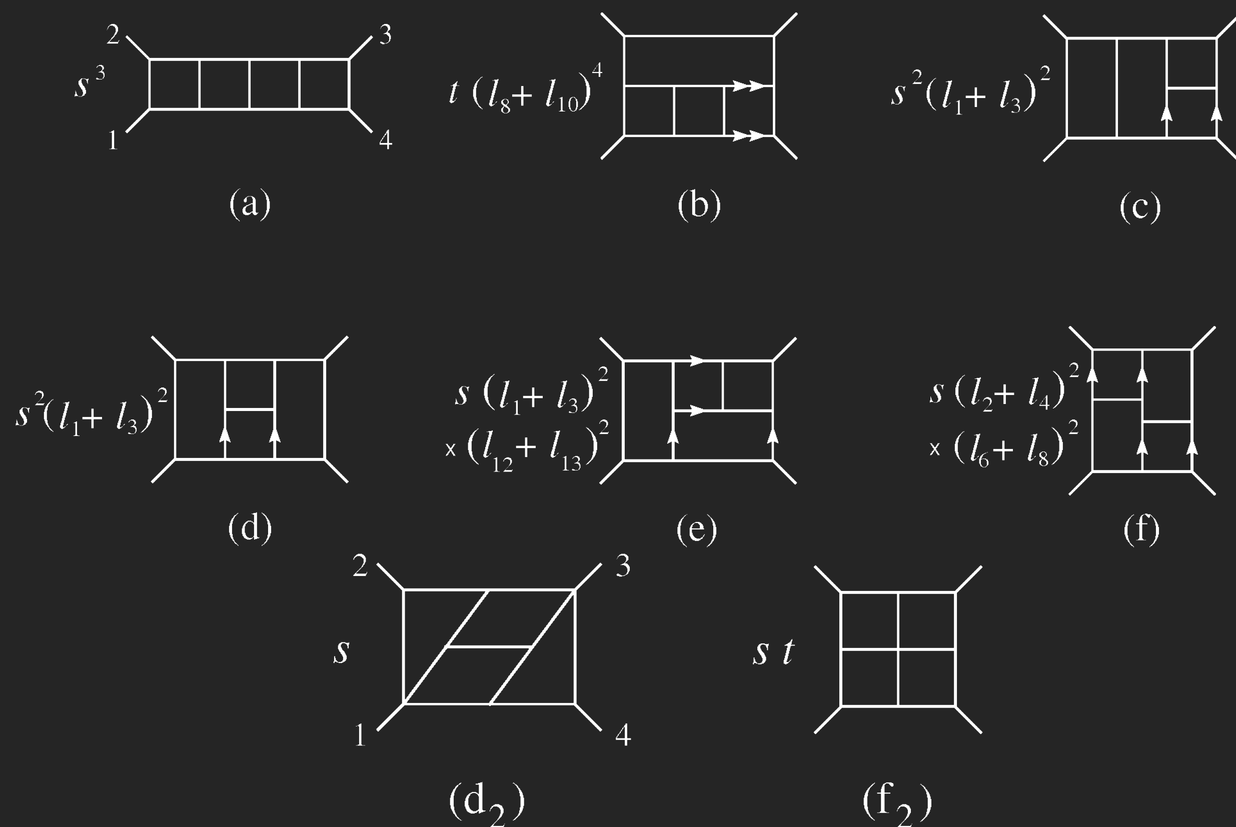


- Not certain that $D = 4$ cuts are sufficient
- Tedious to compute all the scalar & scalar-fermion amplitudes for the three-particle cuts

- Instead, do the cut algebra in $D = 10$ dimensions, for an $\mathcal{N} = 1$ supermultiplet
- Formal polarization vectors/fermion wavefunctions, but no need to compute scalars explicitly
- Distinguish spin dimension from momentum dimension

Integrals in the Amplitude

- 8 integrals present
- 6 given by ‘rung rule’; 2 are new

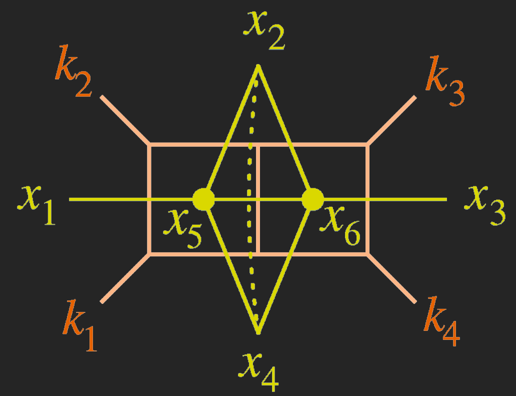


- UV divergent in $D = 11/2$ (vs 7, 6 for $L = 2, 3$)

Conformal Properties

- Consider candidate integrals with external legs taken off shell
- Require that they be conformally invariant
- Easiest to analyze using dual diagrams

Drummond, Henn, Smirnov, & Sokatchev (2006)



- Require that they be no worse than logarithmically divergent
- ⇒ 10 *pseudo-conformal* integrals, including all 8 that contribute to amplitude ⇒ Johansson's talk

Computing Higher-Loop Integrals

- Could use Laporta algorithm (as implemented in AIR or similar) to derive differential equations

- Mellin–Barnes approach developed by Smirnov & automated by Czakon is more direct

Mellin–Barnes Technique

- Introduce Feynman parameters (best choice is still an art) & perform loop integrals

- Use identity

$$\frac{1}{(X+Y)^\lambda} = \int_{\beta-i\infty}^{\beta+i\infty} \frac{Y^z}{X^{\lambda+z}} \frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)} \frac{dz}{2\pi i}$$

  to create an m-fold representation

- Singularities are hiding in Γ functions

Example

$$F^{(\mathrm{a})}(a_1,\ldots,a_{13};s,t;\epsilon)=\frac{e^{4\epsilon\gamma}\,(-1)^a(-s)^{8-a-4\epsilon}}{\prod_{j=2,5,7,9,11,12,13}\Gamma(a_j)\Gamma(4-a_{9,11,12,13}-2\epsilon)}$$

$$\times\frac{1}{(2\pi i)^{11}}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty}\prod_{j=1}^{11}\mathrm{d}z_j\left(\frac{t}{s}\right)^{z_7}\frac{\Gamma(2-a_{9,12,13}-\epsilon-z_{1,2})\Gamma(2-a_{9,11,12}-\epsilon-z_{1,3})}{\Gamma(a_{10}-z_2)\Gamma(a_8-z_3)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

$$\times\frac{\Gamma(a_9+z_{1,2,3})\Gamma(a_{9,11,12,13}-2+\epsilon+z_{1,2,3})\Gamma(z_{10}-z_4)\Gamma(z_4-z_1)}{\Gamma(4-a_{5,8,10}-2\epsilon+z_{1,2,3})\Gamma(4-a_{4,6,7}-2\epsilon+z_{4,5,6})\Gamma(4-a_{1,2,3}-2\epsilon+z_{8,9,10})}$$

$$\times\frac{\Gamma(2-a_5-a_8-\epsilon+z_1+z_3-z_4-z_6)\Gamma(a_{5,9,10}-2+\epsilon-z_{1,2,3}+z_{4,5,6})}{\Gamma(a_3-z_9)\Gamma(a_1-z_8)}$$

$$\times\Gamma(a_{12}+z_1)\Gamma(a_2+z_7)\Gamma(z_7-z_{10})\Gamma(2-a_{6,7}-\epsilon-z_{8,10}+z_{4,5})\Gamma(2-a_{1,2}-\epsilon+z_{8,10}-z_7)$$

$$\times\Gamma(2-a_{4,7}-\epsilon-z_{9,10}+z_{4,6})\Gamma(a_{1,2,3}-2+\epsilon+z_7-z_{8,9,10})\Gamma(2-a_{2,3}-\epsilon+z_{9,10}-z_7)$$

$$\times\Gamma(a_7+z_{8,9,10})\Gamma(2-a_{5,10}-\epsilon+z_{1,2}-z_{4,5})$$

$$\times\Gamma(a_5+z_4+z_5+z_6)\Gamma(a_{4,6,7}-2+\epsilon-z_{4,5,6}+z_{8,9,10})\prod_{j=2,3,5,6,7,8,9}\Gamma(-z_j)$$

- Move contours to expose these singularities (all poles in ^{2})
- Expand Γ functions to obtain Laurent expansion with functions of invariants as coefficients
- Done automatically by *Mathematica* package MB (Czakon)
- Compute integrals numerically or analytically

Result

$$\hat{a} \equiv \frac{g^2 N_c}{8\pi^2} = \frac{N_c \alpha_s}{2\pi}$$

$$\gamma(\hat{a}) = \hat{a} - \frac{\pi^2}{6}\,\hat{a}^2 + \frac{11}{180}\pi^4\,\hat{a}^3 - \left(\frac{73}{2520}\,\pi^6 - (1+r)\zeta_3^2\right)\hat{a}^4 + \mathcal{O}(\hat{a}^5)$$

$$r = -2.028 \pm 0.036$$

Extrapolation to Strong Coupling

- Use KLV approach

Kotikov, Lipatov & Velizhanin (2003)

- Constrain $f_0 \sim \sqrt{\hat{a}}$ at large $\hat{a}$: solve

$$\hat{a}^n = \sum_{r=n}^{2n} C_r \, [\tilde{f}_0(\hat{a})]^r \, ,$$

- Predict two leading strong-coupling coefficients

- Known strong-coupling expansion

$$f_0 = \sqrt{\frac{\hat{a}}{2}} - \frac{3\ln 2}{4\pi} + \mathcal{O}(\hat{a}^{-1/2})$$
$$\approx \sqrt{\frac{\hat{a}}{2}} - 0.16547670011448 + \mathcal{O}(\hat{a}^{-1/2})\,.$$

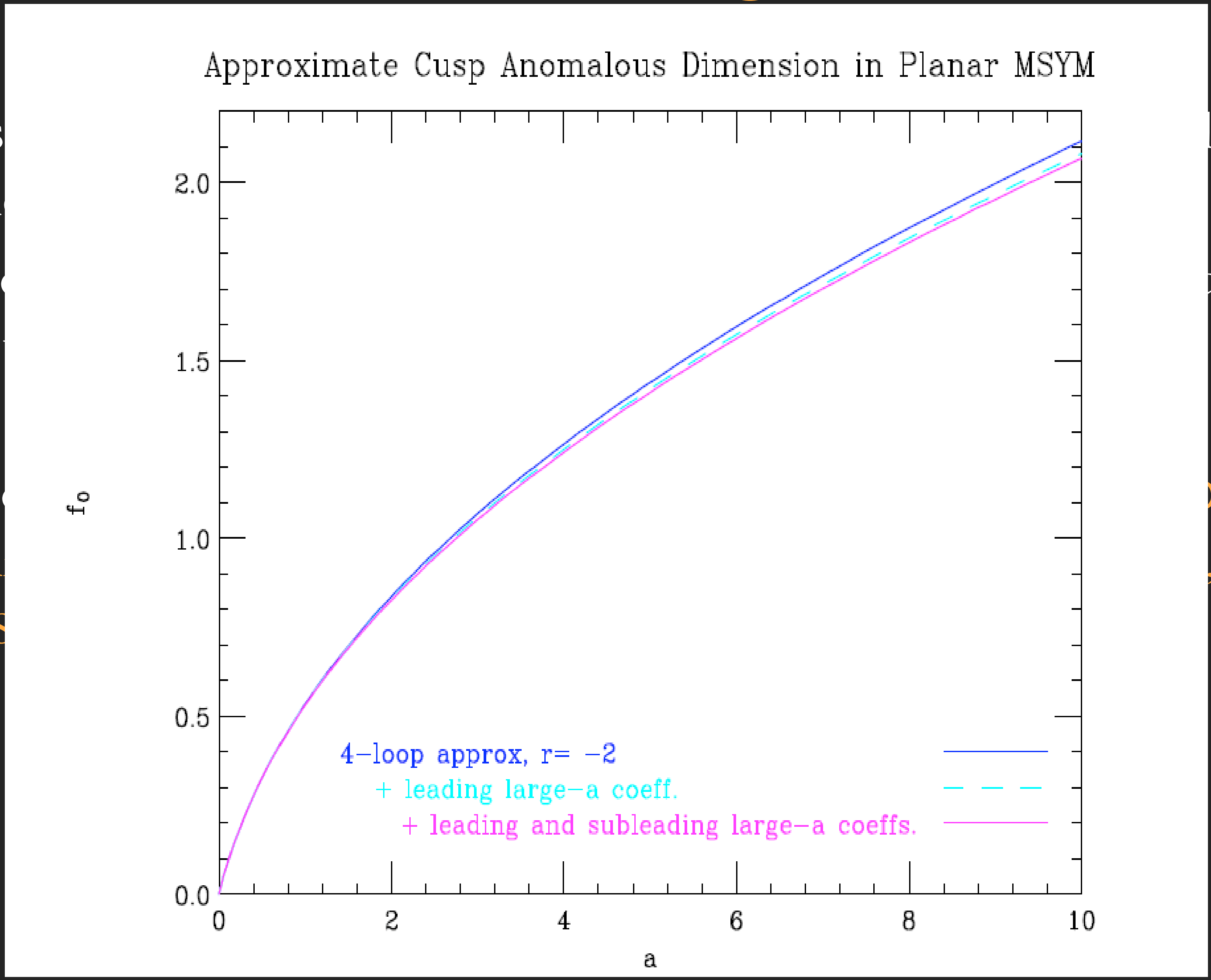
Gubser, Klebanov, Polyakov; Kruczenski; Makeenko (2002)

Frolov & Tseytlin (2002)

- Two loops predicts leading coefficient to 10%, subleading to factor of 2
- Four loop value predicts leading coefficient to 2.6%, subleading to 5%!

Flyover Region

- Us… …ling on…
- Cr… …urate to …
- In … …signs in … …AK, Sm… …, Eden, & S…



Conclusions and Perspectives

"The time has come," the Walrus said, "To talk of many things: Of AdS–and CFT–and scaling-limits– Of spin-chains–and N=4– And whether N=8 is truly finite– And what this is all good for."

- $\mathcal{N}=4$ can be a guide to uncovering more structure in gauge theories and understanding the strong-coupling regime in QCD
- $\mathcal{N}=8$ supergravity can illuminate the same questions
- Unitarity is the method of choice for performing the explicit calculations needed to make progress and ultimately answer the questions ⇒ Bern, Dixon, and Ita's talks on $\mathcal{N}=8$
- What is the connection of the iteration relation to integrability or other structures?