Off-shell Extended Supersymmetry in Harmonic Superspace

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1 Introduction

Supersymmetry leads to a remarkable improvement of the UV behavior of QFT and in some case to UV finiteness. The deep understanding of this phenomenon requires, in my opinion, an off-shell formulation of the theory with manifest linear supersymmetry. More than 30 years after the invention of SUSY this problem has not yet been solved in all possible cases, in particular, for $N = 8$ supergravity.

Here I review the off-shell formulations of two theories with extended SUSY, the $N = 2$ hypermultiplet and $N = 3$ SYM. Both use harmonic superspace with additional bosonic coordinates on a coset of the R-symmetry group. They share the feature of having infinite sets of auxiliary fields coming from the harmonic expansion. I also propose a generalization for $N = 4$ SYM which is based on a twistor transform in 5 dimensions.

2 The $N = 2$ hypermultiplet

2.1 The $N = 2$ hypermultiplet on shell

The $N = 2$ complex hypermultiplet is described by the superfield $q^i(x, \theta^{\alpha i}, \bar\theta_{\dot\alpha i})$ subject to the on-shell conditions (Sohnius)

$$D_{\alpha\, i} q^j = \tfrac{1}{2}\delta_i^j\, D_{\alpha\, k} q^k\ , \qquad \bar D_{\dot\alpha}^{(i} q^{j)} = 0 \qquad \Rightarrow$$

$$q^i = f^i(x) + \theta^{\alpha\, i}\psi_\alpha(x) + \bar\theta_{\dot\alpha\, j}\epsilon^{ij}\bar\kappa^{\dot\alpha}(x) + \text{der. terms}$$

$$\partial\!\!\!/\bar\kappa = \partial\!\!\!/\psi = \partial\cdot\partial f^i = 0$$

No-go theorem: this form of the HM does not exist off shell with a finite set of auxiliary fields (counting argument, Rocek&Siegel, Stelle)

Way out: introduce an infinite set of auxiliary fields obtained by extending the superspace by a complex compact manifold of bosonic variables (Galperin, Ivanov, Kalytsin, Ogievetsky, ES)

$$S^2 = \frac{SU(2)}{U(1)}$$

where $SU(2)$ is the automorphism group of $N = 2$ SUSY.

Harmonic realisation of the coset S^2:

$$u^i_I \in SU(2) \;\; :$$

$$u^I_i = \overline{u^i_I}\,, \quad u^i_I u^J_i = \delta^J_I\,, \quad \epsilon_{ij}\epsilon^{IJ}u^i_I u^j_J = 1$$

where $i = 1, 2$ is an $SU(2)$ index and $I = 1, 2$ is a $U(1)$ index (charge)

Project HM constraints with u^i_2, u^1_i:

$$\text{G-analyticity} \qquad D_{\alpha\,2}\, q^1 = \bar{D}^1_{\dot\alpha}\, q^1 = 0$$

$$D_{\alpha\,2} = u^i_2 D_{\alpha\,i}\,,\;\; \bar{D}^1_{\dot\alpha} = u^1_i \bar{D}^i_{\dot\alpha}\,,\;\; q^1 = u^1_i q^i$$

G-analyticity (also known as $\frac{1}{2}$ BPS shortening condition) can be solved in an appropriate basis (recall the chiral basis) since

$$\{D_2, D_2\} = \{D_2, \bar{D}^1\} = \{\bar{D}^1, \bar{D}^1\} = 0$$

$$x_A^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + i\theta^{\alpha\, I}\bar{\theta}_I^{\dot{\alpha}}$$

$$D_{\alpha\, 2}\, q^1 = \bar{D}^1_{\dot{\alpha}}\, q^1 = 0 \quad \Rightarrow \quad q^1(x_A, \theta^1, \bar{\theta}_2, u)$$

Harmonic dependence given by harmonic expansion on S^2, e.g.:

$$f^1(u) = f^i u_i^1 + f^{(ijk)} u_i^1 u_j^1 u_k^2 + \ldots$$

where $f^{(ijk\cdots)}$ are irreps of $SU(2)$. Homogeneous action of $U(1) \subset SU(2) \longrightarrow$ function on the coset with fixed $U(1)$ weight.

H-analyticity ($SU(2)$ irreducibility): harmonic derivatives on the coset as generators of $SU(2)$: raising (D_2^1) and lowering (D_1^2) operators and $U(1)$ charge ($D_1^1 = -D_2^2$)

$$[D_2^1, D_1^2] = D_1^1$$

$$D_2^1 u_1^i = u_2^i\,, \;\; D_2^1 u_2^i = 0\,, \quad \text{etc.}$$
$$D_1^1 u_1^i = u_1^i\,, \;\; D_1^1 u_2^i = -u_2^i$$

H-analyticity (highest weight irrep):

$$D_2^1 f^1(u) = 0 \quad \Rightarrow \quad f^1(u) = f^i u_i^1$$

The on-shell HM q^1 satisfies the supersymmetrized version of

$$\text{H-analyticity:} \qquad D_2^1 q^1(x, \theta^1, \bar\theta_2, u) = 0$$

$$\text{with} \;\; D_2^1 = (\partial_u)_2^1 + i\theta^1\sigma^\mu\bar\theta_2 \frac{\partial}{\partial x^\mu}$$

This implies an ultrashort superfield

$$q^1 = f^i(x)u_i^1 + \theta^{\alpha\,1}\psi_\alpha(x) + \bar\theta_{\dot\alpha\,2}\bar\kappa^{\dot\alpha}(x)$$
$$+ i\theta^1\sigma^\mu\bar\theta_2\partial_\mu f^i(x)u_i^2$$

and the field equations for f^i, ψ, $\bar\kappa$.

Conjugation: the G-analytic superspace is closed under the combination of complex conjugation and the antipodal map on S^2:

$$\widetilde{(q^1)} = \tilde{q}_2(x, \theta^1, \bar{\theta}_2, u)$$

satisfying the same G- and H-analyticity conditions

$$D_{\alpha\, 2}\; \tilde{q}_2 = \bar{D}^1_{\dot{\alpha}}\; \tilde{q}_2 = D^1_2\; \tilde{q}_2 = 0$$

Conclusion: the combination of G- and H-analyticity conditions gives an equivalent description of the $N = 2$ HM on shell.

2.2 Going off shell

Original description: H-analyticity is kinematical ($SU(2)$ irreducibility) while G-analyticity is dynamical (field equations). Going to the G-analytic basis allows to switch roles: G-analyticity is treated as kinematical and can be solved while H-analyticity becomes dynamical and should be derived from an off-shell action. The existence of an action is always a small miracle:

$$S_{HM} = \int\; d^4x\; du\; d^2\theta^1\; d^2\bar{\theta}_2\;\; \tilde{q}_2\, D^1_2\, q^1$$

Notice the conservation of $U(1)$ charges.

The action contains an infinite set of auxiliary fields coming from the harmonic expansion on the sphere S^2. The field equation

$$\frac{\delta}{\delta \tilde{q}_2} \Rightarrow D^1_2 \, q^1 = 0$$

eliminates all auxiliary fields and puts the physical ones on shell.

Self-interaction of sigma-model type: add a G-analytic potential term

$$\int d^4x \, du \, d^2\theta^1 \, d^2\bar{\theta}_2 \left[\tilde{q}_2 \, D^1_2 \, q^1 + L^{11}_{22}(q^1, \tilde{q}_2, u) \right]$$

Example: $L^{11}_{22} = \lambda (q^1 \tilde{q}_2)^2$ gives rise to the Taub-NUT metric upon elimination of the auxiliary fields.

Quantization is straightforward in the off-shell theory.

Propagator

$$\langle \tilde{q}_2(p,\theta,u)\; q^1(p,\eta,v)\rangle = \frac{i}{p^2\;(u_2^i\, v_i^1)^3}(D_{\theta\,2})^2(\bar{D}_\theta^1)^2\;(D_{\eta\,2})^2(\bar{D}_\eta^1)^2\delta^8(\theta-\eta)$$

Vertex

$$\int du\; d^2\theta^1\; d^2\bar{\theta}_2$$

Finiteness of 2-d $N=(4,4)$ sigma-models by power counting: Restore all vertices $d^4\theta \;\; \longrightarrow d^8\theta$ and do all θ integrals but one. Resulting term in the effective action

$$\lambda^k \int d^8\theta\, d^2p_1 \ldots d^2p_m\; \delta^2(p_1+\cdots+p_m)$$
$$(D)^n\;[q(p_1)\ldots q(p_m)]I(p_1,\ldots,p_m)$$

Power counting gives dim $[I] = -\frac{n}{2} - 2 < 0$ which ensures UV convergence.

3 $N=3$ Super-Yang-Mills

3.1 On-shell curvature constraints

$N = 3$ SYM has the same field contents as $N = 4$ SYM but the fields are organized in representations of $SU(3)$ instead of $SU(4)$. Thus, the 6 real scalars of $N = 4$ SYM are described by a triplet of complex super-Yang-Mills curvatures $W_{ij} = -W_{ji} = \phi_{ij}(x) + \theta$ terms $(i, j = 1, 2, 3)$ satisfying the constraints:

$$\{\nabla_{\alpha\, i}, \nabla_{\beta\, j}\} = \epsilon_{\alpha\beta} W_{ij}$$
$$\{\bar{\nabla}^i_{\dot{\alpha}}, \bar{\nabla}^j_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{W}^{ij}$$
$$\{\nabla_{\alpha\, i}, \bar{\nabla}^j_{\dot{\beta}}\} = i\delta^j_i \nabla_{\alpha\dot{\beta}}$$

These constraints, like the HM ones, imply field equations.

G-analyticity interpretation: Introduce harmonics on the coset $SU(3)/U(1) \times U(1)$

$$u^i_I \in SU(3) \ :$$

$u^I_i = \overline{u^i_I}\,,\ u^i_I u^J_i = \delta^J_I\ ,\ \epsilon_{ijk}\epsilon^{IJK}u^i_I u^j_J u^k_K = 1$
and project the constraints with u^i_3 and u^1_i to obtain G-zero-curvature conditions:

$$\{\nabla_{\alpha\,3}, \nabla_{\beta\,3}\} = \{\bar{\nabla}^1_{\dot\alpha}, \bar{\nabla}^1_{\dot\beta}\} = \{\nabla_{\alpha\,3}, \bar{\nabla}^1_{\dot\beta}\} = 0$$

These are the integrability conditions for the existence of G-analytic (or 1/3 BPS) superfields

$$\nabla_{\alpha\,3}\Phi = \bar{\nabla}^1_{\dot\alpha}\Phi = 0 \quad \Rightarrow \quad \Phi(x, \theta^{1,2}, \bar\theta_{2,3}, u)$$

in the appropriate G-analytic basis in superspace.

The zero-curvature conditions are equivalent to the original on-shell constraints provided the projected covariant derivatives $\nabla_{\alpha\,I}$, $\bar{\nabla}^I_{\dot\alpha}$ are linear in the harmonics. This is guaranteed by H-analyticity.

The compact coset $SU(3)/U(1)\times U(1)$ has 3 complex dimension with corresponding harmonic derivatives (raising operators of $SU(3)$)

D^I_J, $I < J$ satisfying H-zero-curvature conditions

$$[D^1_2, D^1_3] = [D^1_3, D^2_3] = 0\,, \quad [D^1_2, D^2_3] = D^1_3$$

They commute with the projected spinor derivatives (G/H-zero-curvature conditions):

$$[D^I_J, \nabla_{\alpha\,3}] = [D^I_J, \bar\nabla^1_{\dot\alpha}] = 0\,, \qquad I < J$$

which defines $\nabla_{\alpha\,3}$ (and $\bar\nabla^1_{\dot\alpha}$) as the HWS of the (anti)fundamental irrep of $SU(3)$.

Conclusion: on-shell $N = 3$ SYM is equivalent to a set of zero-curvature conditions in a superspace enhanced with harmonic variables on a coset of $SU(3)$ (Galperin, Ivanov, Kalytsin, Ogievetsky, ES, 1984). This resembles the twistor treatment of self-dual Yang-Mills.

3.2 Twistor transform

Before going off shell we do a twistor transform of the constraints. The G-zero-curvature conditions can be solved in the form of 'pure gauges'

$$\nabla_{\alpha\,3} = e^{-ib} D_{\alpha\,3} e^{ib}\,, \quad \bar{\nabla}^1_{\dot{\alpha}} = e^{-ib} \bar{D}^1_{\dot{\alpha}} e^{ib}$$

by introducing a gauge bridge $b(x, \theta, \bar{\theta}, u)$. The twistor transform makes the projected spinor covariant derivatives flat. Instead, the harmonic derivatives D^I_J now acquire connections

$$\nabla^I_J \equiv D^I_J + iV^I_J = e^{ib} D^I_J e^{-ib}\,, \qquad I < J$$

From the mixed G/H-zero-curvature conditions we deduce that the new harmonic connections V^I_J must be G-analytic:

$$V^I_J(x, \theta^{1,2}, \bar{\theta}_{2,3}, u)$$

What remains unsolved are the H-zero-curvature conditions

$$
\begin{aligned}
[\nabla^1_2, \nabla^1_3] = 0 &\leftrightarrow F^{11}_{23} = 0 \\
[\nabla^1_3, \nabla^2_3] = 0 &\leftrightarrow F^{12}_{33} = 0 \\
[\nabla^1_2, \nabla^2_3] = \nabla^1_3 &\leftrightarrow F^{12}_{23} = 0
\end{aligned}
$$

Previously trivial, now they become equivalent to the field equations of $N = 3$ SYM.

3.3 Off-shell action

The crucial observation is that three zero-curvature conditions can be derived from a Chern-Simons-type action provided we can find a supermeasure matching the properties of the CS form ($U(1) \times U(1)$ charges and vanishing dimension). Remarkably, it exists:

$$
S_{N=3} = \int d^4x \, du \, d^2\theta^1 \, d^2\theta^2 \, d^2\bar{\theta}_2 \, d^2\bar{\theta}_3
$$

$$
Tr \left[V^1_2 \, F^{12}_{33} + V^1_3 \, F^{12}_{23} + V^2_3 \, F^{11}_{23} \right]
$$

This action contains infinite sets of auxiliary and of gauge degrees of freedom. Fixing the

WZ gauge and eliminating all auxiliary fields, one finds the standard on-shell action of $N = 3$ SYM.

Grassi & van Nieuwenhuizen have obtained the $N = 3$ SYM action from Witten's open string field theory action by dimensional reduction from 10 to 4 dimensions. The 10-d pure spinors factorize into 4-d spinors and the harmonics u^i_I.

Quantization has been studied by Delduc & McCabe. The Feynman rules resemble those of the $N = 2$ HM and one can do power counting which explains why the theory is finite.

4 Towards an off-shell $N = 4$ SYM theory

Attempts to reproduce the same mechanism in the case of the full $N = 4$ SYM in $d = 4$ have failed. A reformulation of the on-shell constraints as zero-curvature conditions and the subsequent twistor transform are possible, but no suitable measure for the corresponding CS form exists.

An alternative (ES, 1988) could be provided by the Euclidean version of $N = 4$ SYM in $d = 5$. In this case the Lorentz and R symmetry groups are the same, $SO(5) \times SO(5)$. The on-shell constraints have a symmetric form

$$\{\nabla_{\alpha\, i}, \nabla_{\beta\, j}\} = \Omega_{ij}\nabla_{\alpha\beta} + \Omega_{\alpha\beta}W_{ij}$$

where $\alpha, i = 1, 2, 3, 4$ are $USp(4)$ spinor indices and Ω are symplectic matrices.

Zero-curvature conditions can be obtained by projecting with harmonics on the coset

$$\frac{SO(5)}{SO(2)\times SO(3)} \sim \frac{USp(4)}{U(1)\times SU(2)}$$

Despite the apparent equivalence between Lorentz and R-symmetry indices, the harmonic projection must be done on the Lorentz indices, $\nabla_i^a = u^{\alpha\, a}\nabla_{\alpha\, i}$ with $a = 1, 2$ being $SU(2)$ indices, so that

$$\{\nabla_i^a, \nabla_j^b\} = \Omega_{ij}\epsilon^{ab}\nabla_x\,, \qquad [\nabla_i^a, \nabla_x] = 0$$

(the alternative would leave the non-vanishing projected curvature $\epsilon^{ab}u_a^i u_b^j W_{ij}$). Note that the Lorentz harmonic variables are close relatives of pure spinors which are being used for covariant quantization of the superstring by Berkovits.

The twistor transform (this time even closer to the original notion) replaces these constraints by H-zero-curvature conditions

$$[\nabla^{ab}, \nabla^{cd}] = 0$$

involving the 3 covariant derivatives on the harmonic coset $\nabla^{ab} = \nabla^{ba} = D^{ab} + V^{ab}$. The harmonic gauge connections are G-analytic:

$$V^{ab}(x^+, x^{cd}, \theta^c_i, u)$$

Here $x^\pm, x^{cd}$ are Lorentz covariant light-cone coordinates; G-analyticity now involves independence on x^- as well.

Most remarkably, there exists a suitable G-analytic measure for the harmonic CS form which allows to find an off-shell action:

$$S = \int du\, dx^+\, d^3x^{ab}\, d^8\theta^a_i$$

$$\epsilon_{(ab)(cd)(ef)} Tr \left[V^{ab} D^{cd} V^{ef} + \frac{2}{3} V^{ab} V^{cd} V^{ef} \right]$$

Notice that the coupling constant has become dimensionless and that the space-time measure is 4- instead of 5-dimensional. This seems to indicate an effective dimensional reduction from 5 to 4 dimensions, so the underlying theory after the twistor transform might be $N = 4$ $d = 4$ SYM.

More work is needed to understand some subtle points of this unusual formulation. If successful, it might open up a new door towards $N = 8$ supergravity and, who knows, might give us the key to its finiteness?