# Counterterms Revisited:
## Searching for True UV miracles

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UCLA Workshop on ultraviolet properties of N=8 Supergravity

### Some Litterature

Howe + Stelle,
"Ultraviolet Divergences in Higher Dimensional Supersymmetric Yang-Mills Theories"
Phys. Lett. 137 B (1984) 175

Howe + Stelle
"Supersymmetry counterterms revisited"
Phys. Lett. B554 (2003) 190. hep-th/0211279

Drummond, Heslop, Howe and Kerstan
"Integral invariants in N=4 SYM and the effective action for coincident D-branes"
hep-th/0305202

## Maximal SYM theory in higher dimensions

In order to investigate whether there any trully "miraculous" UV cancellations (i.e. cancellations not understandable by counterterm & non-renormalization theorem analysis) it is useful to go to dimensions $D>4$ where maximal SYM eventually diverges. The question is at what loop order $L$ this occurs.

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One early challenge was the $D=6$, $L=2$ maximal SYM calculation of Marcus & Sagnotti, who showed cancellation at this level despite the apparent availability of an allowable counterterm built from the hypermultiplet sector of the theory.

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## SYM in $D=5$ and $D=6$

In $D=5$, $N_D = 1_5$ (i.e. 8-supercharge) SYM has an $USp(2)$ superalgebra automorphism symmetry, while the $N_D = 2_5$ theory (16sc) has $USp(4)$ automorphism symmetry. The superspace spinor coordinates satisfy an $USp(2N)$ pseudo-Majorana condition

$$\bar{\Theta}^{\alpha}{}_{i} = C^{\alpha\beta}\,\Theta_{\beta}{}^{j}\,\Omega_{ji}$$

$\alpha = 1..4 \qquad i = 1\ldots 2N$

$C^{\alpha\beta}$: charge conjugation matrix
$\Omega_{ij}$: $USp(2N)$ invariant (antisym)

For $N_D = 1_5$ SYM, the SYM field strength multiplet is a real scalar superfield $W = \bar{W}$

For $N_D = 2_5$ SYM, the field strength is a pseudoreal $\underset{\sim}{5}$ of $USp(4)$: $\quad W^{ij} = \Omega^{ik}\Omega^{jl}\bar{W}_{kl}$

D=6

In $N_D = 1_6$ (8sc.) supersymmetry, the spinor coordinates are chiral (e.g. left-handed) and satisfy an $SU(2)$ pseudo-Majorana condition $\theta^\alpha_{\;i} = \bar{\theta}^{\alpha j} \epsilon_{ji}$ $\alpha = 1, \dots, 4$ $i = 1, 2$

The $N_D = 1_6$ SYM field strength is a spinor superfield $W^\alpha_{\;i}$

In $N_D = 2_6$ (16sc.) supersymmetry, the additional $\theta$'s are right-handed $\theta_\alpha^{\;i'} = \epsilon^{i'j'} \bar{\theta}_{\alpha j'}$ $\alpha = 1, \dots, 4$ $i' = 1, 2$

while their automorphism symmetry indices $i'$ correspond to a second $SU(2)$, so the overall automorphism symmetry for $N_D = 2_6$ is $SU(2) \times SU(2)$.

The $N_D = 2_6$ SYM field strength superfield is a Lorentz scalar pseudoreal $\underset{\sim}{4}$ under $SU(2) \times SU(2)$

$$W^i_{\;j'} = \epsilon^{ik} \epsilon_{j'l'} \bar{W}_k^{\;l'}$$

- In order to build an off-shell action for the $N_D = 2_6$ theory, the best one can do with finite sets of auxiliary fields is to formulate it in terms of $N_D = 1_6$ superfields. (8 supercharges)
- In addition to the $N_D = 1_6$ SYM sector of the $2_6$ theory, one needs to have a hypermultiplet sector $L$, $L^{ij}$, $L^{ijkl}$ lifted from the off-shell relaxed hypermultiplet in D=4.

## Counterterms

Consider first $N_D = 1_6$ SYM. The non-renormalization theorem allows a generic $L$-loop counterterm

$$\Delta S^L = g^{2(L-1)} \int d^6x\, d^8\theta\, \Delta\mathcal{J}^{(L)}$$

where schematically

$$\Delta\mathcal{J}^{(L)} = D^{4L-3q}\,(gW)^q$$

• For $L=2$, the only possibility is of the form $\Delta\mathcal{J}^{(2)} = D^2 W^2$. However such terms under $\int d^6x\, d^8\theta$ integrate to zero when the equations of motion are used.

$\Rightarrow$ No $L=2$ candidate counterterm for $1_6$ SYM.

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Now consider $2_6$ SYM on-shell. Here, there does appear to be an $L=2$ candidate that does not vanish on-shell:

$$\Delta S^{(2)}_{N_D=2_6} = g^6 \int d^6x\, d^4\theta^{ijkl}\, d^4\theta_{i'j'k'l'}\, \mathrm{tr}\left(W_i{}^{i'} W_j{}^{j'} W_k{}^{k'} W_l{}^{l'}\right)$$

• Indeed, decomposing this into $N_D = 1_6$ superfields, one finds an acceptable $\int d^6x\, d^8\theta\, f(L, L^{ij}, L^{ijkl})$ term in the $D=6$ hypermultiplet sector.

– this is the counterterm that Marcus & Sagnotti were surprized not to find in their $D=6$, $L=2$ calculation.

• However, recall that $N_D = 1_6$ SYM does not have an $L=2$ candidate counterterm.

– what one finds when one decomposes $\Delta S^{(2)}_{2_6}$ into $N_D = 1_6$ superfields is an $N_D = 1_6$ SYM part that <u>cannot</u> be written as a full $\int d^6x\, d^8\theta\, f(A_\alpha{}^i)$ integral in accordance with the non-renormalization theorem.

– So $1_6$ SYM Feynman rules shoot down the $2_6$ candidate.

$\Rightarrow$ Importance of <u>both</u> $N_D$ non.ren. thm. <u>and</u> full on-shell $N_D$ supersymmetry.

Now consider the situation for maximal SYM in $D=5$. For the full $N_D = 2_5$ supersymmetry, we have an on-shell invariant corresponding to $L=4$ loops:

$$\int d^5x\, d^8\theta^{(ijkl)}_{(mnpq)}\, \mathrm{tr}\, W_{(i}{}^{(m} W_j{}^n W_k{}^p W_{l)}{}^{q)} \qquad i,m = 1,\dots,4$$

USp(4) indices

• This seems a similar structure to the one we just disqualified at $L=2$ in $D=6$.

— indeed, both the $D=6$, $L=2$ and $D=5$, $L=4$ candidates reduce to the same $N_D = 4_4$ invariant

$$\int d^4x\, d^4\theta^{ij,mn}\, d^4\bar\theta^{pq,rs}\, \mathrm{tr}\,(WWWW)_{ij,mn,pq,rs}$$

$\underset{\sim}{20}$ $\quad$ $\underset{\sim}{20}$ $\quad$ $\underset{\sim}{105}$ rep. of SU(4)

where the $4_4$ field strength $W_{ij} = -W_{ji}$ satisfies the on-shell constraints
$6$ of SU(4)

$$\nabla_{\alpha i} W_{jk} = \nabla_{\alpha [i} W_{jk]} \quad , \quad \bar\nabla_{\dot\alpha}{}^i W_{jk} = -\tfrac{2}{3}\delta^i_{[j} \bar\nabla_{\dot\alpha}{}^l W_{k]l}$$

• However, there is a difference between the $D=6$ and $D=5$ cases upon reduction to the 8-supercharge off-shell supersymmetry:

— under $N_D = 1_5$, the $1_5$ SYM field strength is a real scalar superfield $W = \bar W$, which has power-counting dimension $1$ as opposed to $3/2$ for the $1_6$ field strength $W^{\alpha}{}_i$, which is a spinor.

— the different power-counting dimension gives rise to generic $1_5$ counterterms $g^{2(L-1)} \int d^5x\, d^8\theta\, D^{2L-2q} (gW)^q$ and for $L=4$ we have a candidate

$$g^6 \int d^5x\, d^8\theta\, (gW)^4$$

allowed by the nonrenormalization theorem and nonvanishing on shell. $\Longrightarrow$ apparently viable $L=4$ candidate in $D=5$!

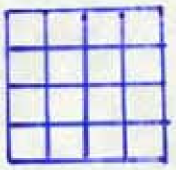
## Supergravity analogue

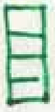
The $N_D = 2_5$ or $4_4$ SYM $\int d^8\theta \, \mathrm{tr}\,(W^4)$ invariants can be compared to the $N_D = 8_4$ $L=3$ maximal supergravity candidate counterterm

Kallosh
Howe,
Stelle &
Townsend

$$\int d^4x \, (d^{16}\theta)_R \, (W^4)_R$$

(lowest order part under linearized $N_D = 8_4$ susy)

R:  232848 rep. of SU(8)

where the field-strength superfield $W_{ijkl} = W_{[ijkl]}$
(containing the 70 scalars at the $\theta = \bar{\theta} = 0$ level)
 $\underset{\sim}{70}$ of SU(8)

satisfies the on-shell constraints

$$D_{\alpha i} W_{jklm} = D_{\alpha [i} W_{jklm]}$$

$$\bar{D}_{\dot{\alpha}}{}^{i} W_{jklm} = -\frac{4}{5} \delta^{i}_{[j} \bar{D}_{\dot{\alpha}}{}^{n} W_{klm]n}$$

## Some predictions from standard superspace Feynman rules

Consider maximal super Yang-Mills (eg $N=4$, $d=4$) in various dimensions of spacetime

- have a formalism with off-shell $M=2$ supersymmetry or, alternatively, a light-cone formulation ($M=\frac{N}{2}$ either way)

Max. Super Yang-Mills expectations for first divergences, standard Feynman rules

| Dimension | 10 | 8 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|
| loop $\ell$ | 1 | 1 | 2 | 3 | 4 | $\infty$ |
| gen. form | $\partial^2 F^4$ | $F^4$ | $\partial^2 F^4$ | $\partial^2 F^4$ | $F^4$ | finite |

Maximal supergravity (e.g. $N=8$, $d=4$) has in principle (although not fully worked out) an off-shell $M=4$ formulation giving the corresponding

Max. Supergravity expectations for first divergences, standard Feynman rules

| Dimension | 11 | 10 | 8 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|---|
| loop $\ell$ | 2 | 1 | 1 | 2 | 2 | 2 | 3 |
| gen. form | $\partial^6 R^4$ | $\partial^2 R^4$ | $R^4$ | $\partial^6 R^4$ | $\partial^2 R^4$ | $R^4$ | $R^4$ |

## $BD^2KPR$ predictions for first divergences

Max SYM divergence expectations: cutting rules

| Dimension | 10 | 8 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|
| loop $\ell$ | 1 | 1 | 2 | 3 | $\boxed{6}$ | $\infty$ |
| gen. form | $\partial^2 F^4$ | $F^4$ | $\partial^2 F^4$ | $\partial^2 F^4$ | $\partial^2 F^4$ | finite |

Max SG divergence expectations: cutting rules

| Dimension | 11 | 10 | 8 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|---|
| loop $\ell$ | 2 | 1 | 1 | 2 | $\boxed{3}$ | $\boxed{4}$ | $\boxed{5}$ |
| gen. form | $\partial^6 R^4$ | $\partial^2 R^4$ | $R^4$ | $\partial^4 R^4$ | $\partial^6 R^4$ | $\partial^6 R^4$ | $\partial^6 R^4$ |

Differences from standard Feynman rule expectations boxed in red.

What's going on?

## A likely irrelevant observation

Howe, Petrini, Stelle

Both the $D=5$, $L=4$ max. SYM and the $D=4$, $L=3$ max. supergravity candidates are built from field strength superfields whose lowest components are scalars
($\boxminus = \underset{\sim}{6}$ of $SU(4)$ for SYM ; $\boxminus = \underset{\sim}{70}$ of $SU(8)$ for supergravity)

- If one could treat these counterterms as invariants of the highest dimensional oxidations of the corresponding theories ($4_4$ SYM $\to$ $1_{10}$ SYM ; $8_4$ SG $\to$ $1_{11}$ SG)

  then these invariants could be ruled out because the scalars would be absorbed into the higher dimensional gauge fields or veilbeins, which must appear inside component field strengths / curvatures in order to respect background gauge invariance.
- Demanding that the scalars be treated this way even in $D=5$ (max. SYM) / $D=4$ (max. SG) while continuing to respect the other restrictions of the off-shell linearly realizable $M_0 = 1_5 / 4_4$ supersymmetry would yield the $BD^2KPR$ predictions for first divergences.
- However, there is no good reason to expect the scalars to be treated this way.

## For SYM, $\mathcal{Z}=4$ however

Starting from helicity $+1$ and applying $N_D=3_4$ lowering operators for massless states, one creates the primitive $N=3$ massless supermultiplet of states:

(1 helicity $+1$, 3 helicity $+\frac{1}{2}$, 3 helicity 0, 1 helicity $-\frac{1}{2}$)

- this needs to be combined with the PCT Conjugate multiplet

  (1 helicity $+\frac{1}{2}$, 3 helicity 0, 3 helicity $-\frac{1}{2}$, 1 helicity $-1$)

  in order to obtain a multiplet of states that can occur in a field theory.

- the net multiplet

(1 helicity $+1$, 4 helicity $+\frac{1}{2}$, 6 helicity 0, 4 helicity $-\frac{1}{2}$, 1 helicity $-1$)

  is precisely the same as that of $N_D=4_4$ SYM

- resulting theory has in fact 16 supercharges, SU(4) automorphism symmetry.

- but this suggests an off-shell realization with $N_D=3_4$ supersymmetry.

- however, there is no such formulation with a finite number of auxiliary fields for the $N_D=4_4$ theory

## N=4 SYM in N=3 harmonic superspace

GIKOS: Galperin, Ivanov, Kalitzin, Ogievetsky, Sokatchev

Consider a superspace with extra bosonic coordinates $(x^{\alpha\dot\alpha}, \theta^{\alpha}{}_{i}, \bar\theta^{\dot\alpha i}, u_i^{(a,b)})$ $\quad x^{\alpha\dot\alpha} = (\sigma_m)^{\alpha\dot\alpha} x^m$

where $u_i^{(a,b)}$ is a coordinate in $SU(3)/U(1)\times U(1)$ "harmonic"

$i$: $SU(3)$ index $\quad (a,b)$: $U(1)\times U(1)$ charges

- One can form
$$\theta^{(a,b)\alpha} = \bar u^{(a,b)i}\,\theta_{\alpha i}$$
$$\bar\theta^{(a,b)\dot\alpha} = u_i{}^{(a,b)}\,\bar\theta^{\dot\alpha i}$$

- then there exists an <u>analytic</u> subspace
$$\left(x_A^{\alpha\dot\alpha},\ \theta^{(1,-1)\alpha},\ \theta^{(0,2)\alpha},\ \bar\theta^{(1,1)\dot\alpha},\ \bar\theta^{(-1,1)\dot\alpha}\right)$$
8 out of 12 $\theta$'s included
$$x_A^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + 2i\left(\theta^{(0,2)\alpha}\bar\theta^{(0,-2)\dot\alpha} - \theta^{(-1,-1)\alpha}\bar\theta^{(1,1)\dot\alpha}\right)$$

Then $N_0=4_4$ SYM can be formulated in this off-shell superspace using analytic connection superfields for the harmonic derivatives:
$$\mathcal{D}^{(\pm1,3)} = D^{(\pm1,3)} + iV^{(\pm1,3)}$$
$$\mathcal{D}^{(2,0)} = D^{(2,0)} + iV^{(2,0)}$$

The GIKOS action is
$$\begin{aligned} I = \int d\zeta_A^{(-2,-6)}\,du\ \mathrm{Tr}\Big( & V^{(2,0)}\left(D^{(1,3)}V^{(-1,3)} - D^{(-1,3)}V^{(1,3)}\right) \\ & - V^{(-1,3)}\left(D^{(1,3)}V^{(2,0)} - D^{(2,0)}V^{(1,3)}\right) \\ & + V^{(1,3)}\left(D^{(-1,3)}V^{(2,0)} - D^{(2,0)}V^{(-1,3)}\right) \\ & - (V^{(1,3)})^2 + 2i\,V^{(1,3)}\left[V^{(-1,3)}, V^{(2,0)}\right]\Big) \end{aligned}$$

- Expansion in $u$ harmonic coordinates gives an <u>infinite</u> set of auxiliary fields.

## Resolution of the SYM discrepancy

Howe
Stelle

- One can perform calculations using harmonic superspace, with basically similar non-renormalization results to the standard Feynman rules.
  (full superspace integrals, no prepotentials)
  — e.g. finiteness of $D=2$ (2,0) $\sigma$-models can be shown this way Sokatchev, Stelle
- N=3 Feynman rules: F. Delduc & J. McCabe
  Applying $N=3$ (12 supercharge) harmonic superspace power counting to the maximal SYM theory reproduces precisely the $BD^2KPR$ results: first $D=5$ divergence at 6 loops.

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For supergravity, 7=8, analogously

Is there an analogous harmonic superspace construction ? Unknown.

However, supposition that there exists an $N=6$ harmonic S.S. formulation gives the present $BD^2KPR$ results (would correspond to KLT: $SG = (SYM)^2$ ).

Supposition of an $N=7$ formulation pushes the order for the first $D=4$ divergence up to 6 loops.

Not my bets

Ignoring all we know about the possibilities for linearly realized $M_4$ supersymmetry and demanding full $8_4$ $\int d^4x\, d^{32}\theta$ integrals over $f(E_M{}^A)$ would push the first divergence to 7 loops.

Demanding in addition that the scalars be treated as if absorbed into $D=11$ veilbeins & gauge fields would push the first divergence to 9 loops.

## On-shell structure of the $\int \partial^2 F^4$ counterterm $\begin{matrix} D=5 \\ L=6 \end{matrix}$

Drummond, Heslop, Howe & Kerstan

- Look at $D=4$ version:

There is a full superspace invariant

$$\Delta = \int d^4x \, d^{16}\theta \, \mathrm{tr}\left(W_{ij} \bar{W}^{ij}\right)$$

which can be expanded using gauge-covariant superspace derivatives $\nabla_{\alpha i} = D_{\alpha i} - igA_{\alpha i}$

— in the case of an Abelian group, the integrand superfield $K = \mathrm{tr}(W_{ij} \bar{W}^{ij})$ (the Konishi superfield) satisfies $D_{ij} K = 0$ $\left(\underset{\text{case}}{\text{Abelian}}\right)$ so the integral $\Delta$ vanishes for Abelian groups.

- By the same token, the $\int \partial^6 F^2$ part of the invariant is absent for non-Abelian Yang-Mills groups.

— however, anticommutators of $\nabla_{\alpha i}$ / commutators of $A_\mu$ produce non-vanishing terms with 4 or 5 field strengths in the non-Abelian case:

$$\begin{aligned}
\int \mathrm{tr}\Big( & 6 F_\alpha{}^\beta F_\beta{}^\gamma F_\gamma{}^\alpha \bar{F}_{\dot\alpha}{}^{\dot\beta} \bar{F}_{\dot\beta}{}^{\dot\alpha} - 2 F_\alpha{}^\beta F_\beta{}^\gamma \bar{F}_{\dot\alpha}{}^{\dot\beta} F_\gamma{}^\alpha \bar{F}_{\dot\beta}{}^{\dot\alpha} \\
& - 6 \bar{F}_{\dot\alpha}{}^{\dot\beta} \bar{F}_{\dot\beta}{}^{\dot\gamma} \bar{F}_{\dot\gamma}{}^{\dot\alpha} F_\alpha{}^\beta F_\beta{}^\alpha + 2 \bar{F}_{\dot\alpha}{}^{\dot\beta} \bar{F}_{\dot\beta}{}^{\dot\gamma} F_\alpha{}^\beta \bar{F}_{\dot\gamma}{}^{\dot\alpha} F_\beta{}^\alpha \\
& + \bar{F}_{\dot\alpha}{}^{\dot\beta} \bar{F}_{\dot\beta}{}^{\dot\alpha} \nabla_{\gamma\dot\gamma} F_\alpha{}^\beta \nabla^{\gamma\dot\gamma} F_\beta{}^\alpha + F_\alpha{}^\beta \bar{F}_{\dot\alpha}{}^{\dot\beta} \nabla_{\gamma\dot\gamma} F_\beta{}^\alpha \nabla^{\gamma\dot\gamma} \bar{F}_{\dot\beta}{}^{\dot\alpha} \\
& + F_\alpha{}^\beta \bar{F}_{\dot\alpha}{}^{\dot\beta} \nabla_{\gamma\dot\gamma} \bar{F}_{\dot\beta}{}^{\dot\alpha} \nabla^{\gamma\dot\gamma} F_\beta{}^\alpha + \bar{F}_{\dot\alpha}{}^{\dot\beta} F_\alpha{}^\beta \nabla_{\gamma\dot\gamma} F_\beta{}^\alpha \nabla^{\gamma\dot\gamma} \bar{F}_{\dot\beta}{}^{\dot\alpha} \\
& + \bar{F}_{\dot\alpha}{}^{\dot\beta} F_\alpha{}^\beta \nabla_{\gamma\dot\gamma} \bar{F}_{\dot\beta}{}^{\dot\alpha} \nabla^{\gamma\dot\gamma} F_\beta{}^\alpha + F_\alpha{}^\beta F_\beta{}^\alpha \nabla_{\gamma\dot\gamma} \bar{F}_{\dot\alpha}{}^{\dot\beta} \nabla^{\gamma\dot\gamma} \bar{F}_{\dot\beta}{}^{\dot\alpha} \Big)
\end{aligned}$$

where $F_{\alpha\dot\alpha,\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta} F_{\alpha\beta} - \epsilon_{\alpha\beta} \bar{F}_{\dot\alpha\dot\beta}$

- Note how the spatial derivatives are contracted together, in agreement with $f(s,t,u)\,F^4$ string amplitude form.