

COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE
AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS
SEPTEMBER 21 AND 22, 1995

Important - please read carefully

The exam (6 hours) is in two parts:

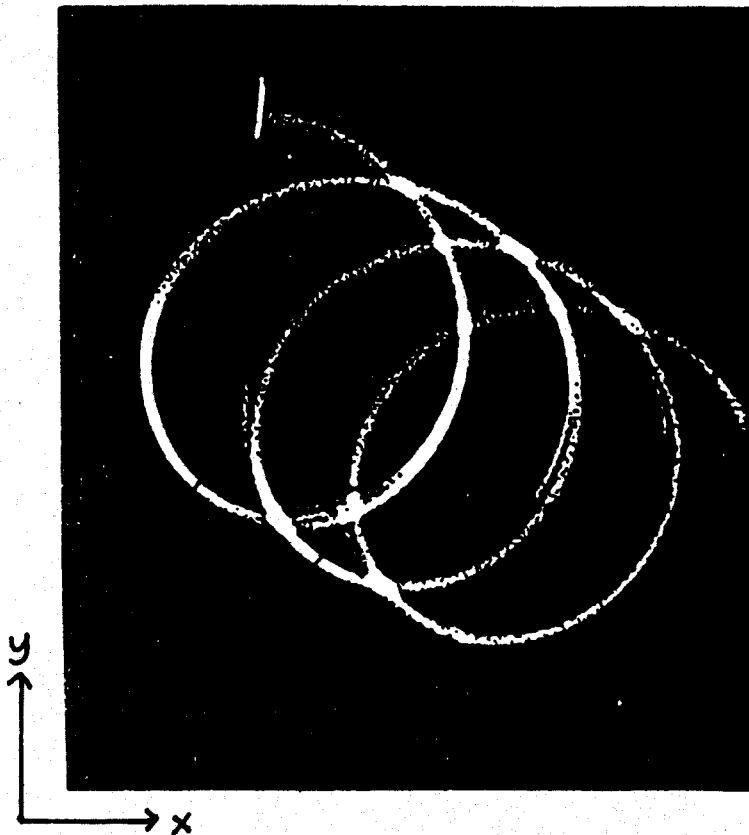
- Part 1 Electromagnetic Theory, and Thermal and Statistical Physics
and General Physics.
- Sept. 21 Do 6 out of 8 E & M problems, both Thermal and General
Physics problems.
- 9:00 - 12:00 This part will be collected at the end of 3 hours.
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- Part 2 Quantum Mechanics, and Thermal and Statistical Physics.
- Sept. 22 Do 3 Thermal & Stat. problems & 5 out of 8 Quantum problems.
- 9:00 - 12:00 This part will be collected at the end of 3 hours.
1. This is a closed book exam.
 2. Work each problem on a separate page.
 3. Print your name and problem # on EACH and EVERY page.
(Note: Pages without names may not be counted.)
 4. Return the problem page as the first page of your
answer.

If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

1. E & M (10 points)

The observed trajectory of an electron beam in a uniform electric and magnetic field is shown below. The beam energy is $\frac{1}{2}mv_1^2 = 100\text{eV}$ and the cyclotron radius is $r_c = 2\text{cm}$. Find the direction and magnitude of \mathbf{E} and \mathbf{B} .

$$(e/m = 1.76 \times 10^{15} \text{ cm}^2/\text{Vs}^2)$$



2. E & M (15 points)

Consider the skin effect in conductors.

- (a) Using Ohm's law and Maxwell's equations derive a differential equation for the current density $\mathbf{J}(\mathbf{r}, t)$.
- (b) Assume a straight cylindrical wire of radius a , uniform conductivity σ and permeability μ . For an ac current at a frequency ω find the radial dependence of the current density. Sketch magnitude and phase versus radius.
- (c) What is the scale length for the current penetration in iron ($\mu = 10^3 \mu_0$, $\sigma = 10^7 \Omega^{-1} \text{ m}^{-1}$) at $f = 60\text{Hz}$?

Hints:

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

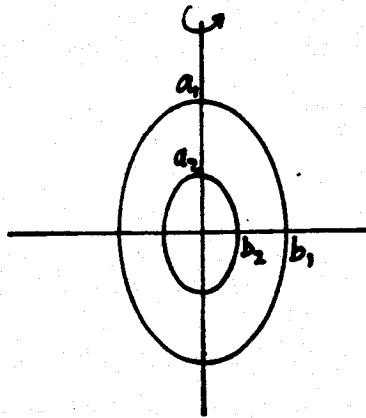
$$J_0(\sqrt{-ix}) = \text{ber}(x) + i \text{bei}(x)$$

3. E & M (15 points)

The capacitance between two confocal elliptical electrodes is given by

$$C = \frac{8\pi\epsilon_0 c}{\ln \left(\frac{a_1 - c}{a_1 + c} \frac{a_2 + c}{a_2 - c} \right)}$$

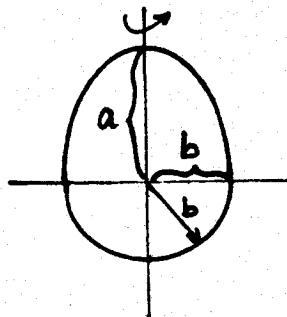
where $c = \sqrt{a_1^2 - b_1^2} = \sqrt{a_2^2 - b_2^2}$ and a, b are semi-major and minor axes, respectively.



Cross-sectional view

Derive the capacitance of the following related capacitors:

- Concentric spheres.
- Single prolate ellipsoid ($a > b$) with respect to a sphere of radius $R \rightarrow \infty$.
- Single elliptical rod ($a \gg b$).
- Flat ellipsoidal disk ($a \ll b$).
- Single electrode consisting of half an ellipsoidal ($a > b$) and half a sphere ($r = b$).



4. E & M (10 points)

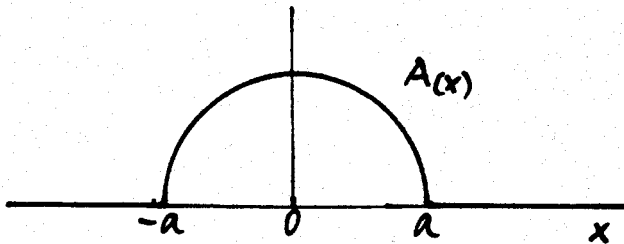
An antenna has a distributed aperture function given by

$$A_{(x)} = \begin{cases} A_0 \sqrt{1 - (x/a)^2} & \text{for } x \leq a \\ 0 & \text{for } x \geq a \end{cases}$$

Find the far-field radiation pattern $A_{(\theta)}$ by Fourier transformation,

$$A_{(\theta)} = \int_{-\infty}^{\infty} A_{(x)} e^{ikx \sin \theta} dx$$

Plot $A_{(\theta)}$ vs. θ and find the angle for the first field null for $ka = 10z_{11} = 38$.



Hints:

$$J_n(z) = \frac{1}{\pi i^n} \int_0^\pi \cos(n\phi) e^{iz \cos \phi} d\phi$$

$$J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z} J_n(z)$$

5. E & M (10 points)

A dielectric separates two large parallel conducting plates of area A . The separation between them is t , and the dielectric constant ϵ varies linearly with position as

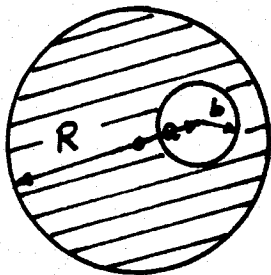
$$\epsilon(x) = 1 + kx \qquad 0 \leq x \leq t.$$

A charge $+Q$ is placed on one plate and $-Q$ on the other.

- (a) Find the electric displacement $D(x)$ ($0 \leq x \leq t$).
- (b) Find the electric field $E(x)$ ($0 \leq x \leq t$).
- (c) Find the voltage V between the plates.
- (d) Find the electrostatic force F between the plates.

6. E & M (10 points)

A long straight conductor carries a current I . As indicated in the figure, the conductor is in the form of a cylinder of radius R with an off-axis cylindrical hole of radius b . Find the magnetic field H in the hole, assuming the current density in the conductor is uniform.



Consider a cubical cavity existing in the region

$$0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1.$$

The "medium" in the cavity is vacuum.

- (a) If the walls of the cavity are perfect conductors, determine the electric and magnetic fields (to within an overall scale and phase) corresponding to the lowest frequency mode with $E_y = E_z = 0$.
- (b) Find the time average electromagnetic energy in the cavity corresponding to the mode in part (a).
- (c) If the walls of the cavity have large, but finite, conductivity σ , magnetic permeability μ and dielectric constant ϵ , obtain the expression for the Q of the cavity corresponding to the (approximate) mode in part (a).

Hints:

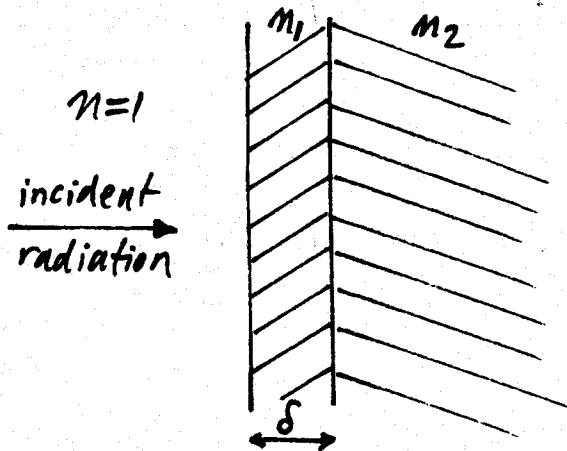
1. If Q of a cavity is $Q = \omega \frac{\text{Average energy stored}}{\text{Average power loss}}$
2. For a good conductor, the spatial dependence of the fields within the walls has the form $e^{-1/\delta} \xi$, where ξ measures the distance into the walls, and δ is the skin depth.

$$\delta = \frac{c}{\sqrt{2\pi\omega\mu\sigma}}$$

8.

E & M (15 points)

Radiation of angular frequency ω is normally incident upon a semi-infinite slab of index of refraction n_2 , which has been coated with a thin layer of material with index of refraction n_1 . The thickness of this layer is δ . The magnetic permeabilities of both the slab and the thin layer have their vacuum value.



- (a) Find a general expression for the ratio of reflected to incident intensities as a function of ω , n , n_2 , and δ .
- (b) Given n_2 , find a set of values for n_1 and δ for which the reflected intensity at frequency ω is zero.

9. Thermodynamics and Statistical Mechanics

The equation linking the magnetization, M , of a set of N magnetic moments and the magnitude of the applied field, H , is

$$M = \mu N \frac{H}{kT \sqrt{1 + (H/m_0 kT)^2}}$$

where the quantities μ and m_0 are constants characteristic to the set of moments. When $H = M = 0$, the heat capacity of the set of moments is given by $C = Nc_0T^4$. Again, c_0 is a physical constant.

- (a) What is the Gibbs magnetic energy, $G(N, T, H)$, of the system?
- (b) Does this system violate the third law of thermodynamics? Justify your answer with a quantitative calculation.

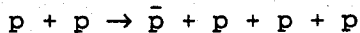
Hints:

$$dU = TdS + HdM + mdN + \text{whatever else;}$$

$$\int^x \frac{dy}{\sqrt{1+y}} = 2\sqrt{1+x} + \text{constant}$$

10. General Physics (15 points)

- (a) Consider the reaction



in which two protons collide to form an antiproton and three protons. If one of the initial protons is at rest, find the minimum energy required of the other initial proton for the reaction to occur. Give answer in terms of Mc^2 , where M = proton mass.

- (b) A Λ^0 particle lives for a time of 2.4×10^{-10} sec in its own rest system. If a Λ^0 particle is created in the laboratory with a velocity $V = \frac{1}{2} C$, how far does it travel before decaying?
(C = velocity of light)

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Part 2 Quantum Mechanics, and Thermal and Statistical Physics.

Sept. 22 Do 8 out of 8 problems.

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1. Thermodynamics I (10 points)

Suppose there are n atoms of an ideal gas per unit volume at temperature T , with mean free path $l = vt$ (v is the mean thermal velocity and t is the average time e between collisions).

- (a) Suppose an ideal gas is confined to a container with a small temperature difference between the ends, so that heat flows from one end to the other. Find an expression for the thermal conductivity of the gas, in terms of the specific heat and the average temperature of the gas. The thermal conductivity is the coefficient relating the temperature difference to the heat flow. You can approximate the average velocity as constant over length scales of the order of the mean free path.
- (b) Explain why the thermal conductivity is independent of pressure (over a wide range of particle densities) at constant temperature.

2. Thermo. and Stat. Mechanics (10 points)

- (a) Using the grand canonical distribution derive the following expression for the particle number fluctuation:

$$\langle (N - \langle N \rangle)^2 \rangle = k_B T \left(\frac{\partial \langle N \rangle}{\partial \mu} \right)_{T, V}.$$

- (b) Consider a substance that has the equation of state of the form

$$PV = \phi(T)$$

where $\phi(T)$ is some function of T , and with energy E a function of T only, i.e.

$$E = \psi(T).$$

By making use of the first and second law of thermodynamics show that the substance must actually obey the ideal gas equation, i.e.

$$PV = RT$$

with R a constant.

3. Thermo. and Stat. Mechanics (10 points)

A very crude model of an enzyme (e.g. myoglobin or hemoglobin) consist of a system of n binding sites for binding molecules of a substrate substance (e.g. oxygen). We associate with each site a variable μ_i , ($i = 1, 2, \dots, n$) which takes the value $+1$ (-1) if the i -th site is occupied (unoccupied) by a substrate molecule. A configuration of the enzyme molecule is then specified by the set of values $(\mu_1, \mu_2, \dots, \mu_n)$, and will be denoted by $\{\mu\}$. For a particular configuration $\{\mu\}$ then, the number of occupied sites is given by

$$\sum_{i=1}^n \frac{1}{2}(1 + \mu_i).$$

- (a) Assume that the binding sites are independent (non interacting), and let

$$p(+1) = Ce^{-J}, \quad p(-1) = Ce^J,$$

give the probability that a site is occupied ($+1$) or unoccupied (-1), respectively.

- I. What requirements determines C for given J ?
- II. Find the average number of occupied sites N .

- (b) Include now the effects of interactions among binding sites. Assume that only nearest-neighbor sites interact, and take the occupation probability distribution to be of the form

$$P(\{\mu\}) = Z^{-1} \prod_{i=1}^n \exp(-J\mu_i) \exp(U\mu_i\mu_{i+1}).$$

- I. What requirement determines Z ?
- II. Show that the average number of occupied sites is given by

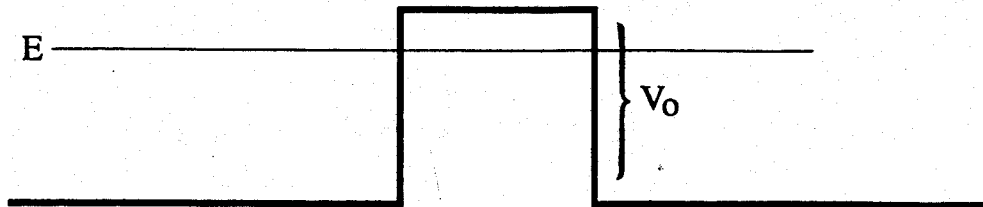
$$N = \frac{n}{2} + \frac{1}{2}M$$

where M is the "magnetization defined by

$$M = -\frac{\partial}{\partial J} (\log Z)$$

4. Quantum Mechanics (10 points)

Consider the once dimensional square potential barrier below.



When the transmitted wave function amplitude ratio T is very small compared to the incident amplitude, T is approximately given by

$$T \approx \exp\left\{-\frac{[2m(V_0 - E)]^{\frac{1}{2}}}{\hbar}\right\}.$$

- (a) Suppose that $V(x)$ is a slowly varying potential. Apply the result above to show that

$$|T|^2 \approx \exp\left\{-2 \int_{x_1}^{x_2} dx \frac{[2m(V_0 - E)]^{\frac{1}{2}}}{\hbar}\right\}.$$

- (b) Now suppose that a metallic object is placed in an electric field, so that near the surface the potential seen by the highest energy electrons varies as

$$\begin{aligned} V(x) &= 0 & (x < 0) \\ V(x) &= V_0 - eEx & (x > a) \end{aligned}$$

The difference $V_0 - E$ represents the work function. The electron sees a triangular potential barrier extending from $x = 0$ to $x = a$, where a is defined by

$$E = V(L) = V_0 - eEa,$$

and therefore

$$L = (V_0 - E)/e = W/eE.$$

An electron of energy E then has a certain probability of penetrating this barrier. Show that the ratio T is given approximately by

$$|T| \approx \exp[-E_0/E],$$

where E_0 is the characteristic field for field emission of electrons. Make an estimate of the characteristic field (i.e., put in some reasonable numbers).

5. Quantum Mechanics (10 points)

Stark Effect of $n = 2$ states for the hydrogen atom. In this problem, an electric field partially lifts the degeneracy of the electronic eigenvalues. The perturbing Hamiltonian is given by

$$H_1 = eEz.$$

The unperturbed wave functions are given by Ψ_{nlm} , where

$$\Psi_{200} = (2a_0)^{-3/2} 2[1 - r/2a_0]e^{-r/2a_0} Y_{00} \quad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$\Psi_{211} = (2a_0)^{-3/2} \frac{1}{\sqrt{3}} \left(\frac{r}{a_0} \right) e^{-r/2a_0} Y_{11} \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin \theta$$

- (a) Write down the equations for the corrections to the ground state energy that occur with the application of the perturbing field.
- (b) Evaluate the corrections and make an energy level diagram. Label the levels with the new eigenfunctions (in terms of the unperturbed functions above).

6. Quantum Mechanics (10 points)

A particle of mass m moves in a one-dimensional potential of the form

$$V(x) = V_0[|x|/a]^\alpha$$

with $\alpha > 0$.

- (a) For $\alpha = 2$, this is the harmonic oscillator problem. It is easy to solve this problem exactly. Do not do so, except as a check on your methods (if you wish to). Instead, use the uncertainty principle to estimate the ground-state energy and the spatial extent of the ground-state.
- (b) Now repeat the same calculations as in part (a), but for $\alpha = 1$, and $\alpha = 4$.
- (c) There is a dimensionless parameter, $y = h^2/2ma^2V_0$, in this problem. For $y \gg 1$, the ground-state energy is an increasing function of α while for $y \ll 1$, it is a decreasing function of α . Explain physically why this is so.

7. Quantum Mechanics (10 points)

Consider as a model of the H^2 molecule, a pair of protons held a fixed distance R apart by a rigid bar which represents the effects of the electrons. Each proton has mass M and spin $\frac{1}{2}$. Ignore all effects of spin-orbit coupling. Therefore, the spins can be arranged into a state of total spin 1 or of total spin 0.

- (a) Show that the ground-state has spin 0 and compute the energy splitting between the spin 0 and the spin 1 ground-states in units of $e = h^2/2MR^2$.
- (b) If the proton were a spin $\frac{1}{2}$ boson (instead of a Fermion) how would the results in (a) be altered?

8. Quantum Mechanics (10 points)

A hydrogen atom is a bound-state of an electron and a proton.

- (a) Ignoring all relativistic effects, such as spin-orbit coupling, hyperfine coupling, etc., what is the degeneracy of the first excited state of the hydrogen atom?
- (b) Some of these degeneracies are accidental, i.e., due to the fact that a number of constants vanish in the limit $c \rightarrow \infty$, and not due to any symmetry of nature. When these other weak couplings are considered, the same states should be classified only in terms of the true symmetries of nature, such as especially full rotational symmetry, i.e., the states should be labeled by their total angular momentum, J . When the effect of the additional very weak relativistic effects are considered, the highly degenerate excited state in (a) should be split into several sets of states with only the degeneracies implied by symmetry. Determine how many distinct energies will result from this splitting, and what the remaining degeneracies of those states will be. Specify also at least the total angular momentum of each of these states. You do not need to specify the form of the perturbing Hamiltonian, nor determine which of these states have lower or higher energies than the others.

9. Quantum Mechanics (10 points)

As the simplest possible model of an atom, we consider an electron of mass m , confined in a harmonic potential of the form $V(r) = (\frac{1}{2})Kr^2$. Compute the polarizability of this atom. Hint: The polarizability is related to the change in the energy of a system when subjected to an electric field to second order in the field strength according to

$$\text{Change in energy} = -(\frac{1}{2})\alpha E^2$$

10. Quantum Mechanics (10 points)

A particle of charge e and mass m moves in constant perpendicular \mathbf{E} and \mathbf{B} fields:

$$\mathbf{E} = (0, 0, E)$$

$$\mathbf{B} = (0, B, 0)$$

- (a) Write down the Schrödinger equation
- (b) Show that, by separation of variables, this reduces to a 1-dimensional problem of a simple harmonic oscillator.
- (c) Find the expectation of the velocity in the x direction:

$$v_x = dx/dt = \frac{1}{i\hbar} [x, H]$$

(in any energy eigenstate).

11. Quantum Mechanics

Two spin-1/2 nuclei fixed in a crystal lattice are subject to a uniform external magnetic field of magnitude B parallel to the z axis. Each nucleus has an intrinsic magnetic moment represented by the operators $K\sigma(1)$ and $K\sigma(2)$ respectively, where K is the constant.

- (a) Neglecting the magnetic dipole interactions between the nuclei, the Hamiltonian of the system is

$$H_1 = H_0 - KB\{\sigma_z(1) + \sigma_z(2)\}$$

where H_0 is independent of spin. Let ψ be the spin-independent eigenfunction corresponding to a given nondegenerate eigenvalue E_0 of H_0 , and suppose the nuclei are in a triplet spin state. Find the corresponding eigenvectors and eigenvalues of H_1 .

- (b) Include now the magnetic dipole interaction between the nuclei given by

$$V = K^2/r^3 [\sigma(1) \cdot \sigma(2) - 3\{\sigma(1) \cdot \hat{\mathbf{r}}\}\{\sigma(2) \cdot \hat{\mathbf{r}}\}].$$

Here \mathbf{r} is the position vector from nucleus 1 to nucleus 2, and $\hat{\mathbf{r}}$ the unit vector in the direction of \mathbf{r} . Thus the total Hamiltonian of the system now becomes $H = H_1 + V$. To first order in perturbation theory, compute the shift in the energy levels found in (a) due to the presence of V .