COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE

AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE

DEPARTMENT OF PHYSICS

Thursday, Sept. 19, and Friday, Sept. 20, 1996

Important — please read carefully.

The exam (6 hours) is in two parts:

Part 1	**Electromagnetic Theory, Statistical Mechanics and Thermodynamics**
Sept 19	8 Problems — **DO 7 OUT OF 8.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.
Part 2	Quantum Mechanics, Thermodynamics and Statistical Mechanics
Sept 20	8 Problems — **DO 7 OUT OF 8.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.

Instructions

1) This is a **closed book** exam and calculators are not to be used.
2) Work each problem on a separate sheet of paper. Use one side only.
3) Print your **name and problem number on EACH AND EVERY** page. (Note: Pages without names may not be counted.)
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5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

1) A metal sphere has a radius R and a charge Q.

(i) Compute the electric part of the Maxwell Stress Tensor

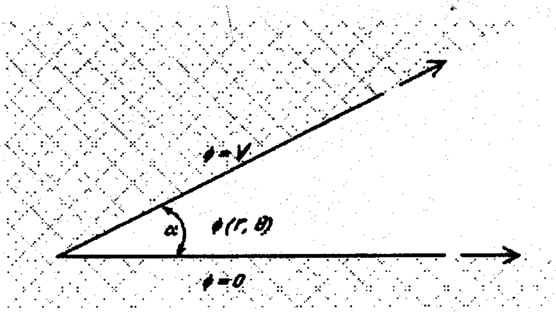
$$T_{ij}(\vec{r}) = \frac{1}{4\pi}\left\{E_i E_j - \frac{1}{2}\vec{E}^2\delta_{ij}\right\}$$

both inside and outside the sphere.

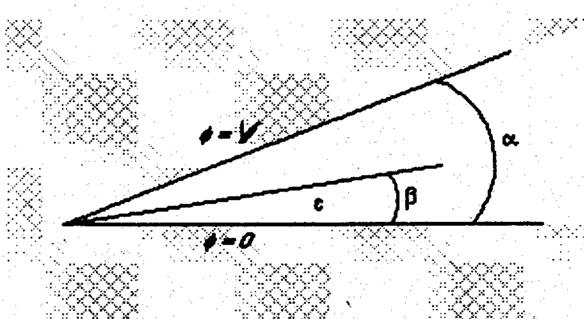
(ii) We now cut the sphere in half along the x-y plane. The two hemispheres repel each other with a force $\vec{F}$. Compute the magnitude and direction of the force $\vec{F}$ on the upper hemisphere using (i).

2) Consider a 2-dimensional (r, θ) electrostatic problem consisting of two infinite wedge-plates making an angle α and held at a potential difference V, as shown below

(a) Find the potential $\phi(r, \theta)$ in the vacuum region between the plates.



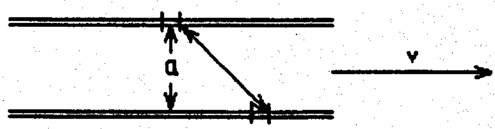
Now insert a dielectric wedge, of dielectric coefficient ϵ and angle $\beta < \alpha$, resting on the bottom plate, as shown below



(b) Find the pressure experienced by the bottom plate at a distance r from the apex.

(c) Find the force per unit volume experienced by the dielectric at a point (r, θ).

3) Two thin, parallel infinitely long, non-conducting rods, a distance a apart, with identical constant charge density λ per unit length in their rest frame, move with a velocity v, not necessarily small compared to the speed of light. Calculate the force per unit length between them in a frame of reference that is at rest, and in a frame of reference moving with the rods, and compare the results. Hint: Use Gauss' Law.



4) An infinitely tall cylinder is placed vertically in a gravitational field with gravitational acceleration g. The bottom of the cylinder is at $z = 0$ and the cylinder cross-sectional area is A. The cylinder contains an ideal gas of N particles of mass m. The gas is in equilibrium.

(i) Using classical statistics, compute the partition function Z.

(ii) Compute the Helmholtz free energy F and the internal energy U.

(iii) Show that the specific heat is $C = \frac{5}{2}Nk_B$. Why is it different from the ideal gas specific heat at constant volume: $C_V = \frac{3}{2}Nk_B$?

5) N charged particles with charge e move in an external potential $\phi_{\text{ext}}(\vec{r})$. Their number density is $n(\vec{r})$.

(i) Express the potential energy U of the system in terms of $n(\vec{r})$.

(ii) By minimizing the free energy $F = U - TS$ with respect to $n(\vec{r})$, keeping N fixed, show that

$$kT \log n(\vec{r}) + e^2 \int \frac{n(\vec{r}')}{|r - r'|} d\vec{r}' + e\,\phi_{\text{ext}}(\vec{r}) = \mu$$

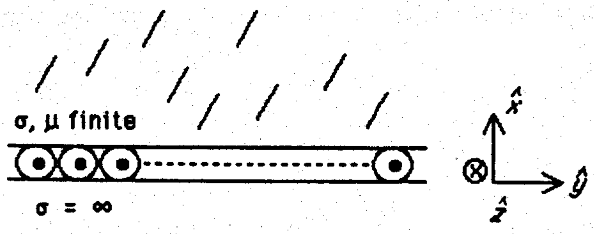
with μ a constant. Use: $S = -k_B \int d\vec{r}\, n(\vec{r})(\log n(\vec{r}) - 1)$.

(iii) Show, using (ii), that:

$$n(\vec{r}) = n_0 e^{-e\phi(\vec{r})/k_B T}$$

with n_0 a constant and $\phi(\vec{r})$ the total electrical potential. Give a physical explanation of this result. (This expression is the basis of Debye-Hückel theory for ionized systems.)

6) A current sheet carrying a surface current $\vec{\lambda} = \lambda_0 \cos\omega t\, \hat{z}$ is sandwiched between a perfect conductor ($\sigma = \infty$) and a material having finite conductivity σ and magnetic permeability μ. The frequency ω is sufficiently low that magnetostatic conditions prevail. λ_0 is a constant, $\hat{z}$ is a unit vector parallel to the interface located at $x = 0$, and t is the time.



(a) Find the appropriate partial differential equation that governs the behavior of the magnetic field **H** for $x > 0$ (above the current sheet). Do *not* solve.

(b) What is the appropriate boundary condition for **H** in this system?

(c) Find the magnetic field **H** at arbitrary distance $x > 0$ at time t.

7) Consider a collection of N non-interacting rigid rotors fixed on a plane (i.e., no center of mass motion and rotation about one axis only) and having moment of inertial I. This system is in thermal equilibrium with a heat bath at temperature T.

(a) Write down the canonical partition function Z_i for a **single** rotor. [Remember that angular momentum is quantized].

(b) Use the Euler-Maclaurin approximate summation formula

$$\sum_{m=0}^{\infty} f_m \approx \int_0^{\infty} dx f(x) + \frac{1}{2}[f(\infty) + f(0)]$$

to obtain the molar specific heat c_v for large temperatures. (Useful to know: $\int_0^{\infty} dx e^{-x^2} = \sqrt{\frac{\pi}{2}}$).

(c) Find the average energy per particle for low temperatures.

8) Consider a spherical capacitor consisting of inner and outer shells of radii a and b respectively. The region between the two shells is filled with a non-magnetic conducting dielectric of conductivity σ and dielectric constant ϵ. The free charge uniformly distributed on the inner and outer shells is Q_f and $-Q_f$ respectively.

(a) Explicitly show that at every point P, where $b > r_p > a$,

$$\frac{\partial \vec{D}}{\partial t} + 4\pi \vec{J} = 0$$

(b) Explicitly show that the total rate of energy dissipation due to Ohmic heating is equal to the rate at which the capacitor is losing electrostatic energy.

(c) Determine the time evolution of the total electrostatic energy.

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9) Consider a spin one-half particle. Show that in the space of the states of a given orbital angular momentum ℓ, the operators

$$\frac{\ell + 1 + \vec{L}\cdot\vec{\sigma}}{2\ell + 1} \qquad \text{and} \qquad \frac{\ell - \vec{L}\cdot\vec{\sigma}}{2\ell + 1}$$

are projection operators onto the states of total angular momentum $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$ respectively.

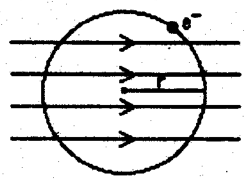
10) A system consists of two distinguishable particles, each with intrinsic spin $\frac{1}{2}$. The spin-spin interaction of the particles is $J\vec{\sigma}_1 \cdot \vec{\sigma}_2$, where J is a constant. An external magnetic field **H** is applied. The magnetic moments of the two particles are $\alpha\sigma_2$, and $\beta\sigma_2$. Find the exact energy eigenvalues of this system. Hint: It is convenient to use a representation where the total spin and its z component are diagonal.

11) An electron with charge e and mass m is confined to move on a circle of radius r. It is perturbed by a uniform electric field F parallel to one of the diameters of the circle.

(a) Find the unperturbed energy levels.

(b) Find the shift to first order in F.

(c) Find the second order shift. Notice in particular the special care needed for the first excited state.



12) A system with unperturbed eigenstates ϕ_n and energies E_n is subject to a time dependent perturbation $H'(t) = \frac{A}{\sqrt{\pi}} e^{-t^2/\tau^2}$ where A is a time independent operator.

a) If initially ($t = -\infty$) the system is in its ground state ϕ_0, show that to first order, the probability amplitude that at time $t = +\infty$ the system will be in its n^{th} state ($m \neq 0$) is

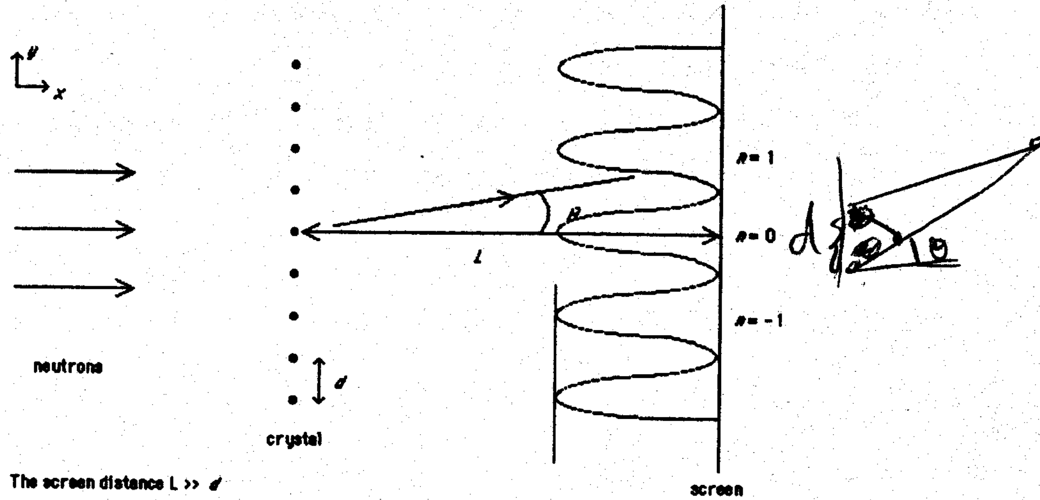
$$C_m = -\frac{i}{\hbar} \langle \phi_m | A | \phi_0 \rangle e^{-(E_0 - E_m)^2 \tau^2 / 4\hbar^2}$$

b) Consider the limit of an impulsive perturbation where $\tau = 0$. Show that the probability P that the system makes any transition out of the ground state is

$$P = \frac{1}{\hbar^2} [\langle \phi_0 | A^2 | \phi_0 \rangle - \langle \phi_0 | A | \phi_0 \rangle^2]$$

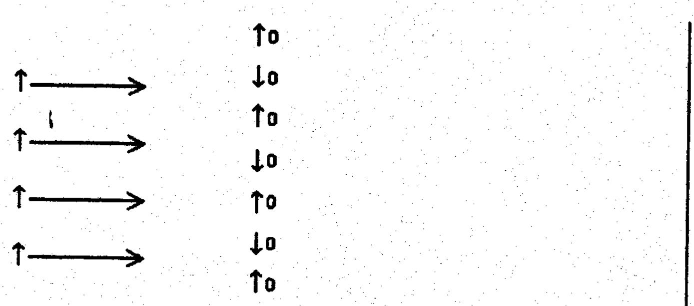
Hint: Find the transition probability to the n^{th} state and sum over all excited states.

13) A beam of neutrons falls on a crystal. The scattered neutrons are detected on a screen and show interference fringes. For simplicity, we treat the crystal as a one-dimensional array of point-scatterers with a spacing d, and the problem as two-dimensional. The incoming wave is a coherent beam $\psi \propto e^{ikx}$.



(i) Show that the angular position θ_n of the nth maximum (see figure) obeys: $d \sin\theta = n\lambda$ where the distance to the screen $L \gg d$.

(ii) Neutrons actually have a spin 1/2. Assume that the incoming neutrons all are polarized along the positive axis. Assume that the atoms also carry a spin 1/2, and that their spin is alternatingly polarized along the positive and negative y axis (a so-called antiferromagnet). During the scattering, the neutron spins interact with the atomic spin. Assume that a neutron scattering from an ↑ atomic spin shift gains a scattering phase-shift of 180° compared to a neutron scattering from an ↓ atomic spin. Compute the angular positions of the diffraction maxima and draw in the figure below.



14) A photon gas in thermal equilibrium is contained within a box of volume V at temperature T.

(a) Use the partition function to find the average number of photons $\bar{n}_r$ in the state having energy E_r.

(b) Find a relationship between the radiation pressure p and the energy density u (i.e., the average energy per unit volume).

(c) If the volume containing the photon gas is decreased *adiabatically* by a factor of 8, what is the final radiation pressure if the initial one is p_0?

15) Four identical particles constrained to move in one dimension are described by the Hamiltonian

$$H = \sum_{i=1}^{4} H_{sp}(x_i)$$

where

$$H_{sp} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

a) Find the energies of the system and identify the eigenfunctions, *treating the particles as distinguishable.*

b) What is the degeneracy of the lowest two levels (still treating the particles as distinguishable).

c) The same as (b) but for states made up of four identical spin zero particles.

d) Describe the ground state if the system consists of four identical spin one-half particles. What is its degeneracy?

16) The meson $K^+(m_{K+} = 493.68$ MeV) and the baryon Λ^0 (with mass $m_{\Lambda^0} = 1115.68$ MeV) are produced in the reaction $p + p \rightarrow p + K^+ + \Lambda^0$ ($m_p = 938.27$ MeV). If one of the protons in the initial state is at rest in the laboratory frame, determine the minimum kinetic energy that is necessary for the other proton to initiate this reaction. Also, at this minimum energy determine the laboratory velocities of the p, K^+ and Λ^0 in the final state.