

**COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE
AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS**

Thursday, September 18, and Friday, September 19, 1997

Important — please read carefully.

The exam (6 hours) is in two parts:

Part 1 Electromagnetic Theory, Statistical Mechanics and Thermodynamics

September 18 7 Problems — **DO ALL PROBLEMS.**

9:00-12:00 This part will be collected at the end of three hours.

Each problem counts for 20 points; the total is 140.

Part 2 Quantum Mechanics, Thermodynamics and Statistical Mechanics

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- 3) Print your name and problem number on **EACH AND EVERY** page. (Note: Pages without names may not be counted.)
- 4) Return the problem page as the first page of your answer.
- 5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

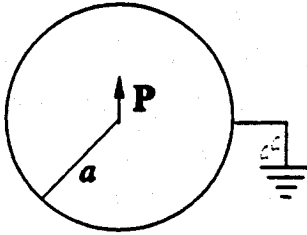
- 1) There is a constant magnetic induction $\mathbf{B} = B\hat{\mathbf{z}}$ throughout a certain region of space.
- (a) Show that the vector potential there can be written in the form $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$. This result is not unique, however. How are different vector potentials for the same field \mathbf{B} related to one another?
- (b) A constant magnetic induction $\mathbf{B} = B\hat{\mathbf{z}}$ is present everywhere except inside a long cylinder of radius a with axis parallel to $\hat{\mathbf{z}}$. The magnetic induction vanishes inside the cylinder. Find a vector potential which is valid $\mathbf{A}(\mathbf{r})$ everywhere. Show explicitly that your result is consistent with Ampere's law.

2) Two long, co-axial (i.e. one inside the other and centered on the same axis), cylindrical conducting surfaces of radii a and b ($> a$) are lowered vertically into a dielectric liquid. A potential difference V is applied between the cylindrical surfaces.

- (a) Will the dielectric liquid rise or fall between the electrodes when the potential difference is applied? Give a detailed explanation of the phenomenon.
- (b) Relate the electric susceptibility χ_e of the liquid to the change in height, h , of the liquid, its density, ρ , the acceleration due to gravity, g , and the other parameters in the problem.

3) An ideal electric dipole \mathbf{P} is located at the center of a hollow sphere of radius a , as in the figure below. The sphere is grounded:

- (a) Find the electrostatic potential Φ at an arbitrary point located within the sphere.
- (b) Deduce what is the energy of interaction between the ideal dipole and the grounded sphere.



4) DNA molecules dissolved in water can be modeled as infinite rods with a radius a ($\approx 10\text{\AA}$) and a charge per unit length λ ($\approx -e/1.7\text{\AA}$).

- (a) Compute the electrostatic potential $\varphi(r)$ as a function of the distance r from the axis of the rod. Let ϵ_W be the dielectric constant of water and assume $\varphi(r=R) = 0$ with $R \gg a$.
- (b) Now add a sufficient number of monovalent, positive "counter-ions" to the cylinder of radius R to neutralize the rod. Compute the density profile $\rho(r)$ of the counter-ion using the Boltzmann distribution. Neglect interaction between the counter-ions. Make sure you normalize $\rho(r)$ properly.
- (c) The normalization of $\rho(r)$ has a mathematical singularity. Compute the temperature where this happens and speculate on the physical interpretation.

5) Show that the magnetic field in a superconductor (*i.e.* a dissipationless conductor with charge carrier density of n , charge e and mass m) obeys the London equation

$$\nabla^2 \mathbf{B} = -\frac{\mathbf{B}}{\lambda^2}$$

A superconducting sphere is placed in a constant external magnetic field \mathbf{B}_0 , show that the maximum value of the field on the surface of the sphere is $(3/2)B_0$.

7) Linearly polarized light of form $E_x(z, t) = E_0 e^{i(kx - \omega t)}$ is incident normally onto a material which has an index of refraction n_r for right-hand circularly polarized light and n_l for left-hand circularly polarized light. Calculate the intensity and polarization of the reflected light.

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8) A two-dimensional oscillator has the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(1 + \delta xy)(x^2 + y^2),$$

where $\hbar = 1$ and $\delta \ll 1$.

- (a) Give the wave functions for the two lowest energies when $\delta = 0$.
- (b) Evaluate the change in the energy levels for $\delta \neq 0$ (to lowest order in δ).

9) Consider the scattering of a beam of low-energy spinless particles by an attractive spherical well of depth V_0 and radius a in three dimensions:

$$V(r) = \begin{cases} -V_0 & \text{for } r < a, \\ 0 & \text{for } r > a. \end{cases}$$

Show that the scattering cross-section is

$$4\pi a^2 \left(\frac{\tan(ka)}{ka} - 1 \right)^2, \quad k \equiv \frac{((E + V_0)2m)^{1/2}}{\hbar}.$$

10) A nonrelativistic particle of mass m is bound in a central potential $V(\mathbf{r}) = V(r)$ that is finite at the origin.

- (a) Show that its wave function in spherical coordinates can be written in the factorized form $\psi(\mathbf{r}) = R(r)Y(\theta, \phi)$ because the kinetic-energy operator $\mathbf{T} = \mathbf{T}_r + \mathbf{T}_L$ can be separated into a radial part and an angular part. Show that the radial part \mathbf{T}_r can be expressed in terms of the radial part \mathbf{p}_r of the linear momentum operator \mathbf{p} , and that

$$\mathbf{p}_r \equiv \frac{1}{2}(\mathbf{e}_r \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{e}_r) = \frac{\hbar}{i} \frac{1}{r} \frac{\partial}{\partial r} r,$$

where $\mathbf{e}_r = \mathbf{r}/r$ is the unit vector along \mathbf{r} .

- (b) Obtain the normalized angular function $Y(\theta, \phi)$ for S -states. Show that the S -state radial function $R(r)$ could have solutions near the origin of the forms

$$R_1(r) = A \quad \text{and} \quad R_2(r) = B/r.$$

where A and B are real constants. Show that both solutions are square-integrable near the origin.

- (c) Show that $R_2(r)$ is not admissible as a solution for the wave function because the radial momentum \mathbf{p}_r is not Hermitian (or self adjoint) for this type of wave function.

11) Consider an atom that has a lowest electronic state 1S_0 and a first excited state 3S_1 with energy ϵ_1 above the ground state. All higher states with energies above the first excited state have energies much greater than kT .

- (a) For a gas of the atoms at temperature T , find the fraction of excited atoms.
- (b) Give the internal energy of a gas of N atoms as function of temperature (i.e. $U(T)$) and compute from this the specific heat $c_V(T)$. Make a plot of $c_V(T)$.

Note:

$$\frac{eV}{k_B} = 1.16 \times 10^4 \text{ }^\circ\text{K} ; \quad \frac{eV}{hc} = 8.05 \times 10^3 \text{ cm}^{-1} .$$

12) Consider N sites placed along a circle with equidistant spacing between sites. Let $\psi_n(t)$ be the amplitude of an electron to be in state $|n\rangle$ occupying the n^{th} site. (Ignore electron spin.) Let $E_0 = \langle n|H|n\rangle$ be the energy of an electron on site n , and $J = -\langle n|H|n+1\rangle$ the "transfer energy".

(a) Show that

$$i\hbar \frac{\partial}{\partial t} \psi_n(t) = E_0 \psi_n(t) - J(\psi_{n+1}(t) + \psi_{n-1}(t)). \quad (1)$$

(b) Look for solutions which are energy eigenstates:

$$\psi_n(t) = a_n e^{-iEt/\hbar} \quad (2)$$

and show

$$Ea_n = E_0 a_n - J(a_{n+1} + a_{n-1}). \quad (3)$$

(c) Prove that (3) is solved by $a_n = e^{ikn}$ and find a relationship between E and k .

(d) From the circular geometry of the problem, find a condition on allowed values of k in terms of N . You now have found the spectrum of energy eigenstates. Give the lowest three energy eigenvalues.

13) Consider a spinning electron described by a (Pauli) spinorial wave function

$$\Psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}.$$

Write down the expressions for the operators of orbital angular momentum \mathbf{L} , spin angular momentum \mathbf{S} and total angular momentum \mathbf{J} .

(b) Let $L^2 = \mathbf{L} \cdot \mathbf{L}$, etc; what are the allowed eigenvalues of L^2 , S^2 , J^2 in this situation?

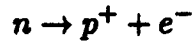
(c) In particular, given an allowed value of L^2 , what are the possible values of J^2 ?

(d) Fix L^2 , S^2 , J^2 . Construct the associated eigenvectors of J_x . Hint: The eigenvectors of L^2 are spherical harmonics. S^2 is a diagonal matrix, the eigenvectors of J^2 are spinors with spherical harmonics as coefficients.

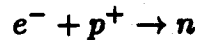
14) A white dwarf star has a mass $M = 2 \times 10^{23}$ grams and a radius $R = 10^8$ cm. You can assume that all electrons in the star are ionized and relativistic (so the electron energy $\epsilon(p) \approx pc$).

(a) Prove that the Fermi energy $\epsilon_F \propto \hbar c(N/V)^{1/3}$ with (N/V) the electron density. Show that the relativistic description is reasonable.

(b) Within a star, a neutron can decay into a proton



with the electron energy $\Delta\epsilon \simeq 0.8 \times 10^6$ eV. The reverse reaction



is possible if the electron energy exceeds $\Delta\epsilon$. (Both neutrons and protons are fermions.) It is found that, as the radius R of the star decreases, the star's protons transform into neutrons. Explain and compute the star's radius R_c for this to start to happen (such stars are called neutron stars).

(c) What is the star's Fermi energy if $R = 0.1 R_c$? [Hint: Are we still in the relativistic regime?]

$$c = 3 \times 10^{10} \text{ cm/sec.}$$

$$\hbar = 10^{-27} \text{ erg.}$$

$$m(\text{electron}) = 9.1 \times 10^{-28} \text{ g} = .51 \text{ MeV}/c^2.$$

$$m(\text{proton}) = m(\text{neutron}) = 1.6 \times 10^{-24} \text{ g} = 940 \text{ MeV}/c^2.$$

$$1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg.}$$