COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE

DEPARTMENT OF PHYSICS

Thursday, March 27, and Friday, March 28, 1997

Important — please read carefully.

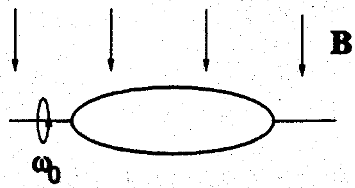
The exam (6 hours) is in two parts:

Part 1	**Electromagnetic Theory, Statistical Mechanics and Thermodynamics**
March 27	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.
Part 2	Quantum Mechanics, Thermodynamics and Statistical Mechanics
March 28	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.

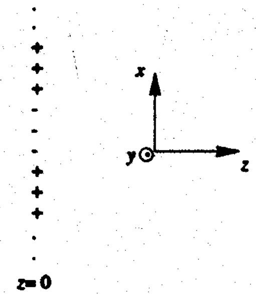
Instructions

1) This is a **closed book** exam and calculators are not to be used.
2) Work each problem on a separate sheet of paper. Use one side only.
3) Print your **name and problem number on EACH AND EVERY** page. (Note: Pages without names may not be counted.)
4) Return the problem page as the first page of your answer.
5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

1) A poorly conducting thin circular ring of electrical conductivity σ, mass m and radius r rotates with no friction about an axis perpendicular to a uniform magnetic field $\mathbf{B}$. It is set in motion with an initial frequency of rotation ω_0. Demonstrate that the frequency of rotation decays in time and calculate the corresponding decay constant τ. [Hint: The moment of inertia of the ring is $mr^2/2$.]



2) A two-dimensional free charge distribution is given by $\sigma_0 \delta(z) \cos kx$, where σ_0 and k are positive constants and $\delta(z)$ is a Dirac delta function. The variables x, y, z refer to the Cartesian coordinate system shown below:



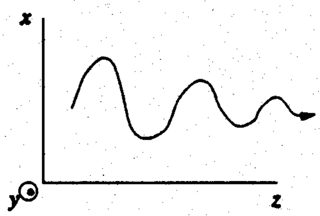
The free charge is embedded inside a crystal dielectric whose dielectric tensor is uniaxial, i.e. $\epsilon_{zz} = \epsilon_{\parallel}$, $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{\perp}$ and all the off-diagonal components are zero.

(a) Write down the Poisson equation for the electrostatic potential ϕ for this system. Use coordinate system shown to label derivatives properly.

(b) Solve for the potential ϕ at a location $z > 0$.

3) A spacecraft measures that the electric field of an electromagnetic wave propagating through the solar wind has the following functional dependence:

$$\mathbf{E}(z,t) = E_0 e^{-\alpha z} \cos(kz - \omega t)\,\hat{x}$$



where E_0, α, k, ω are real constants (positive), (x, y, z) refers to a Cartesian coordinate system and we consider $z > 0$, and t is the time.

(a) Deduce the oscillatory current density in this region of the solar wind.

(b) If the solar wind has a mass density ρ, deduce the acceleration (time-averaged, per unit volume) being exerted by the wave on the solar wind at a location $z > 0$.

4)

(a) Show from general invariance considerations that a charge q cannot radiate if it is not accelerated.

(b) Show that the radiated power P, or energy loss per second, is proportional to q^2a^2, where a is the acceleration if the electromagnetic interaction is weak. Use dimensional analysis to determine the proportionality constant $D = P/q^2a^2$ for your own choice of a system of units. We shall take any additional dimensionless constant to be 1 for the purpose of this problem.

(c) A particle of mass m and charge q is connected to a fixed point in free space by a massless rod of length L. An infinite grounded plane conductor is located at a distance D ($\gg L$) from the point of support. Show that the charge will undergo oscillations about an equilibrium position. Determine the frequency of small oscillations about this equilibrium position. (Assume that the charge moves only in a plane and ignore the effect of radiation on the frequency.)

(d) Estimate the time-averaged radiated power P of this oscillating charge by using the results of Part (b).

5) A binary alloy is composed of N_A atoms of type A and N_B atoms of type B. Over the temperature range of interest the A-type atoms can only be found in the ground state and in an excited state of energy E_0. Similarly the B-type atoms can only occupy the ground state and an excited state having energy $2E_0$. The entire mixture is in thermal equilibrium at temperature T.

(a) Calculate the Helmholtz free energy F of this alloy.

(b) Deduce the heat capacity of this system at temperature T.

6) A system consisting of N atoms with spin $1/2$ is placed in a magnetic field H and is kept at temperature T. Find the entropy, the internal energy, the total magnetic moment, and the specific heat of the system. (Assume the gyro-magnetic ratio is $g = 2$).

7) Consider a very long solenoid that is aligned with the z-axis of an ordinary, Cartesian coordinate system. A constant electric current is passed through the wire.

(a) Use what you know about magnetic field lines to give a plausibility argument why the field outside a very long solenoid is very weak.

(b) Consider the ideal limit of an infinitely long solenoid when the magnetic field $\mathbf{B}$ is vanishingly small outside the solenoid. The vector potential $\mathbf{A}$ in the region outside the solenoid is a solution of the equation $\mathbf{B} = \mathbf{curl}\ \mathbf{A}$. Thus $\mathbf{curl}\ \mathbf{A} = 0$ outside the solenoid. Show that $\mathbf{curl}\ \mathbf{A}$ can satisfy this equation without $\mathbf{A}$ itself being zero: $\mathbf{curl}\ \mathbf{A} = 0$ but $\mathbf{A} \neq 0$.

(c) Suppose that the total magnetic flux inside the solenoid is Φ. Prove that $\mathbf{A} \neq 0$. **Hint:** From the laws of electromagnetism relate the vector potential outside the solenoid to the total flux.

(d) Find a vector potential which is consistent with (c). **Hint:** Use cylindrical symmetry.

COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE
AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS
Thursday, March 27, and Friday, March 28, 1997

Important — please read carefully.

The exam (6 hours) is in two parts:

Part 1	Electromagnetic Theory, Statistical Mechanics and Thermodynamics
March 27	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.
Part 2	**Quantum Mechanics, Thermodynamics and Statistical Mechanics**
March 28	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.

Instructions

1) This is a **closed book** exam and calculators are not to be used.
2) Work each problem on a separate sheet of paper. Use one side only.
3) Print your **name and problem number on EACH AND EVERY** page. (Note: Pages without names may not be counted.)
4) Return the problem page as the first page of your answer.
5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

8) A one-dimensional potential

$$V(x) = \begin{cases} -V_0, & \text{for } x < 0 \\ 0, & \text{for } x > 0 \end{cases}$$

can be used to describe a metal-insulator junction. (V_0 is a positive constant.) Electrons with an energy sufficient to leave the metal (i.e. the $x < 0$ region) may be reflected at the interface ($x = 0$).

(a) Find the reflection coefficient at the surface for an electron with energy $E > 0$.

(b) What is the ratio of the number of the reflected electrons to the number of the transmitted electrons for a junction of $V_0 = 10$ eV and electrons with an energy of 0.1 eV?

9) Consider a two-state quantum mechanical system with energies E_1 and E_2. Suppose it is acted upon by a time-independent perturbation represented by the matrix:

$$\delta V = a \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

where a is small compared to $E_1 - E_2$. If at time $t = 0$, it is in state 1, find the probability that it is in state 2 after a time T.

10) Consider a Schrödinger eigenvalue problem of the form

$$H|i\rangle = E_i|i\rangle, \quad i = 1, 2, \dots ,$$

where H is a self-adjoint Hamiltonian and $E_1, E_2 \dots$ is the set of eigenvalues associated with a complete, discrete set $|1\rangle, |2\rangle, \dots$ of eigenvectors of H.

(a) Denote the inner product of two states ψ, ϕ by $\langle\psi|\phi\rangle$. Prove that

$$\langle i|j\rangle = 0 \text{ if } E_i \neq E_j.$$

(b) Show that, if A is any well defined operator acting on the eigenstates, then the expectation value of $[A, H]$ on any eigenstate of H is zero.

(c) Consider the case of a hydrogen atom. Use the statement in (b), with H the Schrödinger hydrogen atom (i.e., Coulomb interaction) Hamiltonian and A the dilation (scaling) operator

$$A = \sum_{i=1}^{3} x_i \frac{d}{dx_i}$$

to prove that the expectation value of the kinetic energy in a bound state is equal to (the absolute value of) the binding energy and the expectation value of the potential energy is -2 times as much. (The x_i are the three spatial coordinates.)

11) Consider a hydrogen atom

(a) Obtain the normalized wave function and the energy of the ground state by showing that your results satisfy a time-independent Schrödinger equation. You may assume that the electron mass is negligibly small compared to the proton mass.

A hydrogen atom in its ground state is moving with velocity $\mathbf{v}$ in the lab. At time $t = 0$, the proton in the atom is *suddenly* brought to rest during a time τ as the result of a collision with a neutron that does not interact with the electron.

(b) What is the probability that the hydrogen atom will remain in its ground state after the collision? Give your answer first as an integral over $\mathbf{r}$. Simplify your integral by explicitly integrating over all angles.

(c) How sudden must the collision be for your result to be valid? Give your answer in terms of quantities appearing in part (a).

12) A gas that deviates slightly from ideal behavior exhibits an equation of state $pv = RT - a/v$, where p is the pressure, T is the temperature, R is the universal gas constant, a is a constant, and v is the volume per unit mole, i.e. $v = V/\nu$. The system consists of N particles.

(a) Deduce what should be the dependence of the partition function Z on the volume V for this gas.

(b) Use your knowledge of the behavior of an ideal gas to deduce what is the fully normalized (including quantum phase space) partition function Z for this system.

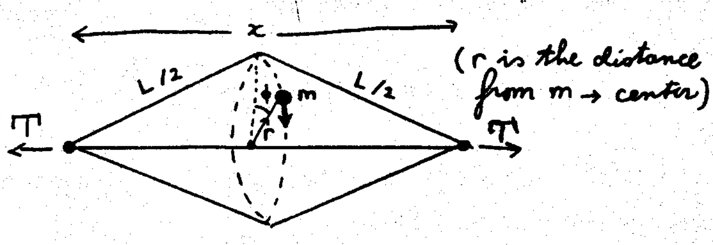
13) A two-dimensional electron gas has area density σ.

(a) Show that the Fermi energy ϵ_F of the system is $\pi\sigma\hbar^2/m$.
(b) What is the density of states?
(c) Show that the chemical potential is given by:

$$\mu(T) = k_B T \ln[\exp(\epsilon_F/k_B T) - 1] \ .$$

(d) Show that the average kinetic energy is $k_B T$ in the high temperature limit ($T \to \infty$) and $\epsilon_F/2$ in the low temperature limit ($T \to 0$).

14) A weight of mass m is fixed to the middle point of a string of length L as shown in the figure. It rotates about an axis joining the string ends whose spacing is x. The system is in contact with its environment at a temperature T. (Ignore gravity and assume that the string does not stretch or vibrate.)



(a) What is the Hamiltonian of this system in terms of the angle ϕ and the canonical momentum p_ϕ?

(b) Compute the thermal average $\langle p_\phi^2 \rangle$.

(c) Show that the tension T acting on the ends of the string required to keep x fixed is independent of m.