COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE
AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS
Thursday, September 24, and Friday, September 25, 1998

PART I: THURSDAY, SEPTEMBER 24

Important — please read carefully.

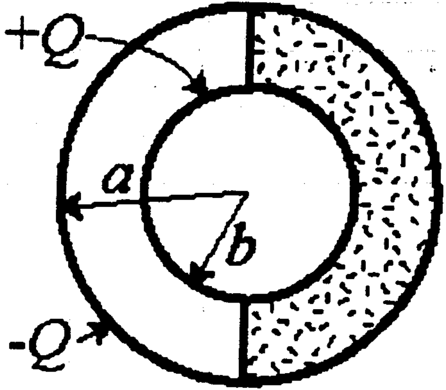
The exam (6 hours) is in two parts:

Part 1	Electromagnetic Theory, Statistical Mechanics and Thermodynamics
September 24	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.
Part 2	Quantum Mechanics, Statistical Mechanics and Thermodynamics
September 25	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.

Instructions

1) This is a **closed book** exam and calculators are not to be used.
2) Work each problem on a separate sheet of paper. Use one side only.
3) Print your **name and problem number on EACH AND EVERY** page. (Note: Pages without names may not be counted.)
4) Return the problem page as the first page of your answer.
5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

1) (E&M) Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric, having a dielectric constant ϵ, as shown in the figure.



a) Find the electric field everywhere between the spheres.

b) Calculate the surface charge distribution on the inner sphere.

c) Calculate the polarization charge density induced on the surface of the dielectric at $r = a$.

2) (E&M) *Paramagnetic sphere.* Consider a sphere of paramagnetic material in an external applied magnetic field $\vec{B}_0 = B_0\hat{z}$. (I.e., as $r \to \infty$, $\vec{B} \to \vec{B}_0$.) Let the sphere have radius R_0 and suppose that inside the sphere, the magnetization $\vec{M}$ satisfies

$$\vec{M}(\vec{B}) = \frac{M_0\,\vec{B}}{B_c + B},$$

where $B = |\vec{B}|$, and where M_0 and B_c are known positive constants which obey $8\pi M_0/3B_c < 1$. Calculate the field inside and outside the sphere. Sketch the field strength at the center of the sphere as a function of B_0. Discuss briefly what happens when $8\pi M_0/3B_c > 1$.

3) (E&M) *Motion of a charge in a light wave.* Consider a charge q that starts from rest at $x = y = z = 0$ at time $t = 0$. The charge is accelerated by a linearly polarized light wave with $\vec{E} = E_0 \cos(kz - \omega t)\, \hat{y}$. Here $\hat{y}$ is the unit vector in the y direction.

a) Calculate the magnetic field $\vec{B}$ of the light wave.

b) Calculate the position of the charge as a function of time, to lowest order in v/c; assume $v/c \ll 1$.

c) Calculate the correction to the position of the charge in the z direction, to one order higher in v/c than in part b).

d) Sketch the motion in the y–z plane with the correction from part c).

4) (E&M) *Radiation from a charge distribution.* In the radiation zone, the vector potential from a current oscillating at frequency ω is

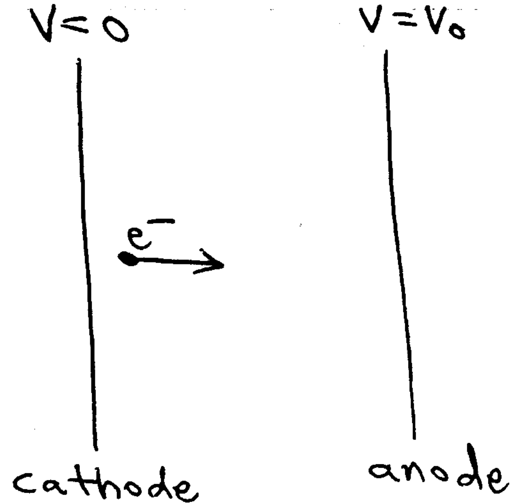
$$\vec{A}_\omega(\vec{r})e^{-i\omega t} = \frac{e^{i(kr-\omega t)}}{cr} \int \vec{J}_\omega(\vec{r}\,')\, e^{-i\vec{k}\cdot\vec{r}\,'}\, d^3\vec{r}\,',$$

where $\omega = kc$ and $\vec{k} = k\hat{r}$ (and $\hat{r} = \vec{r}/r$).

a) Suppose $\vec{J}_\omega(\vec{r}) = J_0(\omega)\ \hat{x}\ e^{-r^2/a^2}$, where $\hat{x}$ is the unit vector in the x direction. Calculate the electric and magnetic fields in the radiation zone.

b) Give an expression for the total power radiated.

5) (E&M) Consider an idealized parallel plate diode, as shown. At equilibrium, electrons are emitted in large numbers with negligible velocity from the cathode (at voltage $V = 0$) and are attracted to the anode (held at constant $V = V_0$). Find the voltage (potential) $V(x)$ and the current density $J(x)$ between the plates at equilibrium, in terms of the electron mass m, charge magnitude e, V_0, d, and x. Hint: At the appropriate place in the solution, assume that the functional form of V is $V(x) \sim x^y$ for some (real) y.



6) (Stat. Mech./Thermo.) Here are the equations of state for an ideal gas:

$$PV = Nk_BT; \quad U = \frac{3}{2}Nk_BT,$$

where P is the pressure of the gas, N is the number of particles, T is the temperature, U is the internal energy, and k_B is Boltzmann's constant. Derive the Helmholtz free energy, $F(N, V, T)$, of the ideal gas. From this quantity, find the entropy as a function of temperature, number of particles, and volume.

7) (Stat. Mech./Thermo.) Consider a collection of N particles, each of which can occupy one of two single-particle states, with energies ϵ_0 and ϵ_1, respectively.

a) In terms of N, U (the total internal energy of the collection of particles), ϵ_0, and ϵ_1, what is the entropy of this system?

b) Rewrite your expression in the form appropriate for $N \gg 1$. (Hint: $\ln(N!) \approx N \ln N - N$.)

c) Obtain an expression for the temperature $T(U, N)$ of this system. What unusual property does this temperature have? Comment on it.

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8) (Quantum Mechanics) A positron has the same mass m as the electron, but the opposite charge. An electron-positron Coulomb-bound state is called "positronium."

Compare and contrast the main features of the hydrogen atom with those of positronium. Assume there are no external electric or magnetic fields, and don't bother to discuss any higher order quantum electrodynamics effects you may know about. Include at least

i) An energy-level diagram of the $n = 1$ and $n = 2$ levels.

ii) The relative sizes of the gross level splittings, the fine-structure, and the hyperfine structure splittings.

iii) The absolute size (e.g. in eV) of the ground-state binding energies.

9) (Quantum Mechanics) An electron is injected into a region where there is a constant magnetic field of magnitude B along the z-axis. At $t = 0$, the direction of the electron's momentum is along the x-axis, perpendicular to the magnetic field, and it is completely polarized so that its spin is along the direction of the beam.

Let θ be the angle between the electron's momentum and the expectation value of its spin. Let g be the gyromagnetic ratio of the electron. At $t = 0$, $\theta = 0$. What is θ as a function of the time t? The time-dependence of the momentum can be computed classically.

10) (Quantum Mechanics) Consider a hydrogen atom with total orbital angular momentum-squared $2\hbar^2$. A measurement of the z-component of orbital angular momentum results in the value $+\hbar$. One then measures the component of orbital angular momentum along the direction $\hat{n}$, where $\hat{n} = (\sin\theta, 0, \cos\theta)$. What are the possible results of the second measurement? What is the probability (as a function of θ) that the result is $+\hbar$?

For an angular momentum, for j=1, in the usual representation:

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad J_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

11) (Quantum Mechanics) *Isotope shift.* Consider a nucleus of radius R with Z protons and N neutrons, with $A = Z + N$. We consider the charge to be spread out uniformly over the volume of the nucleus.

a) Find the potential energy $V(r)$ of an electron a distance r from the center of the nucleus, such that $V(r = \infty) \equiv 0$.

b) Recall $V(r)$ for the hydrogenic atom with point nucleus. We can use perturbation theory to solve the finite-sized-nucleus problem if we take as our perturbation the *difference* between our point-nucleus V and the V in part a). Calculate the first-order correction to the ground state of a hydrogen atom. (Hint: make an obviously good approximation so that you don't have to do any integrals.)

c) Explain qualitatively the *sign* of your result.

The wave function for the ground state of a hydrogenic atom can be written as

$$\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} \exp(-(Zr/a_0)),$$

in terms of standard quantities.

12) (Quantum Mechanics) (After G.L. Squires.) For a certain particle in one dimension in coordinate space, the operator $\hat{A}$ corresponding to the physical quantity A does not commute with the Hamiltonian $\hat{H}$. $\hat{H}$ has two normalized eigenfunctions $u_1(x)$ and $u_2(x)$, with eigenvalues E_1 and E_2, respectively. $\hat{A}$ has two normalized eigenfunctions $w_1(x)$ and $w_2(x)$, with eigenvalues a_1 and a_2, respectively. The eigenfunctions are related by:

$$w_1 = (u_1 + u_2)/\sqrt{2}; \quad \text{and} \quad w_2 = (u_1 - u_2)/\sqrt{2}.$$

Suppose that at t=0, the wave function is $\psi(x, t{=}0) = w_1(x)$. Assuming that no measurements are made, find the expectation value of A for $t > 0$. Your expression should be in a form which is manifestly real, and which makes the frequency of the time dependence clear.

13) (Quantum Mechanics) The binding of molecular hydrogen (H_2) is a simple example of the covalent bond. The mechanism can be simulated by a linear diatomic molecule with a potential energy for an electron given by a pair of Dirac δ-functions of equal strength separated by an interatomic distance $2a$: $V(x) = -A\delta(x-a) - A\delta(x+a)$. Derive the transcendental equations which give the energy eigenvalues for the symmetrical and anti-symmetrical energy eigenstates, in terms of clearly defined symbols. You don't need to solve the equations. Hint: one of the equations to derive is $k(\tanh(ka)+1) = 2mA/\hbar^2$.

14) (Stat. Mech./Thermo.) An ideal monatomic gas of N particles, each of mass m, is in thermal equilibrium at absolute temperature T. The gas is contained in a cubical box of side L, whose top and bottom sides are parallel to the earth's surface. The effect of earth's uniform gravitational field on the particles should be considered, the acceleration due to gravity being g.

a) What is the average kinetic energy of a particle?

b) What is the average potential energy of a particle?