COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE
AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS
Thursday, September 23, and Friday, September 24, 1999

PART I: THURSDAY, SEPTEMBER 23

Important — please read carefully.

The exam (6 hours) is in two parts:

Part 1	Electromagnetic Theory, Statistical Mechanics and Thermodynamics
September 23	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
	Each problem counts for 20 points; the total is 140.
Part 2	Quantum Mechanics, Statistical Mechanics and Thermodynamics
September 24	7 Problems — **DO ALL PROBLEMS.**
9:00-12:00	This part will be collected at the end of three hours.
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Instructions

1) This is a **closed book** exam and calculators are not to be used.
2) Work each problem on a separate sheet of paper. Use one side only.
3) Print your **name and problem number on EACH AND EVERY** page. (Note: Pages without names may not be counted.)
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5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

1) (E&M) *Reflection of light from a conductor.*

a) Suppose a linearly polarized monochromatic electromagnetic wave propagating in the (positive) z direction in a vacuum is incident on a perfect conductor at $z = 0$. Assuming $\vec{E} = 0$ in the conductor, calculate the reflected wave: give both the $\vec{E}$ and $\vec{B}$ fields.

b) Now suppose the conductor is not perfect, but has high conductivity σ, with $\vec{J} = \sigma \vec{E}$. For what values of σ can you ignore the displacement current in Maxwell's equations in the conductor?

c) In the regime where we can ignore the displacement current, calculate the amplitude of the reflected wave.

2) (E&M) Suppose that you have a collection of electrons in a rigid, uniform, ionic background. The electrons obey the following equation of motion in a time-varying electric field:

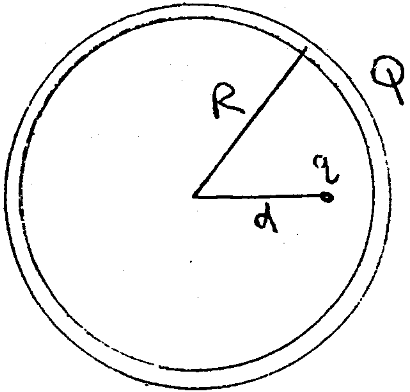
$$m\frac{d^2\vec{r}}{dt^2} + \gamma\frac{d\vec{r}}{dt} = (-e)\vec{E}(t),$$

where m is the electron mass, $(-e)$ the electron charge, and γ is a viscuous drag coefficient. Assume that the number density of electrons is n_0.

a) Find the frequency-dependent dielectric constant, $\epsilon(\omega)$, of this system.

b) Let γ be small. Locate the singularities of $1/\epsilon(\omega)$ in the complex ω plane. Comment on your findings.

3) (E&M) A thin metallic shell of radius R carries a charge Q. A point charge q is introduced into the interior of the shell a distance d from its center. Find the force acting on this point charge.



4) (E&M) *(Relativistic Oscillations)*. Consider the motion of a relativistic charged particle in an electrostatic potential. Let $\vec{E} = -\nabla\phi$ and $q\phi = mc^2 z^2/L^2$ where m is the mass and q the charge.

a) Show that $\gamma = \gamma_0 + z^2/L^2$ where γ_0 is a constant and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

b) Give an expression for the period of oscillation (as an integral). Show that in the non-relativistic limit the period is independent of particle energy.

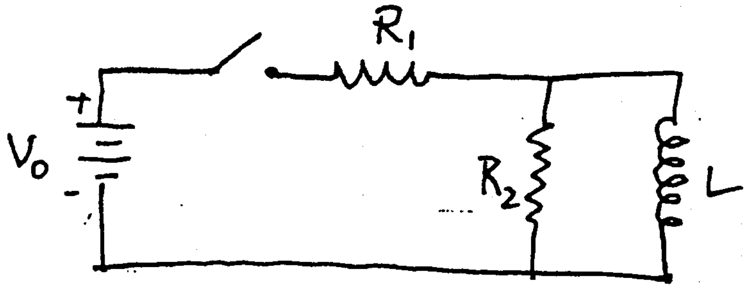
c) Find an expression for the amplitude and period in the ultra-relativisitic limit. Explain your result in physical terms.

5) (E&M)

a) The circuit shown has a battery with $V_0 = 100$ V, inductor $L = 10$ H, and resistors $R_1 = 10\ \Omega$ and $R_2 = 100\ \Omega$.

i) At $t = 0$ the switch is suddenly closed and remains closed for a long time. Find the total energy dissipated in R_2.

ii) The switch is suddenly opened, and remains open a long time. Find the total energy dissipated in R_2 after the switch is opened.



b) (Unrelated to part a).) Can a copper wire be floated in the earth's magnetic field (about 1 gauss) by passing a current through it? The mass density of copper is $\rho_m = 8.9$ g/cm^3 and its resistivity is $\rho_e = 1.7 \times 10^{-8}$ ohm-m. Assume that this is being tried at the equator. How much power per unit volume will need to be supplied to the wire?

6) (Stat. Mech./Thermo.) A certain insulating solid contains N_A non-magnetic atoms and N_B magnetic impurities, each with spin $\frac{3}{2}$. Each impurity spin is free to rotate independently of the rest. There is a very weak spin-phonon interaction which couples the lattice to the magnetic impurities.

a) A magnetic field is applied to the system while it is held at a constant temperature T. The field is strong enough to line up the spins completely. What is the change in entropy as the field is applied?

b) The system is now thermally isolated, and the magnetic field is slowly reduced to zero. Assuming that the heat capacity of the lattice is given by its classical limit, compute the final temperature of the system.

7) (Stat. Mech./Thermo.) Vacancies are thermally-created defects (missing atoms) in crystals. In a crystal, both the atoms and vacancies are found to be on the periodical lattice sites. Assume there are N atoms and N_v vacancies in a crystal with $N + N_\mathrm{v}$ sites, and write an expression for the entropy of the system. If the temperature of the solid is held at T and the energy cost to create a vacancy is E_v, find the free energy. Show that the equilibrium number of vacancies N_v is $N \exp(-E_\mathrm{v}/k_B T)$ when $N_\mathrm{v} \ll N$. (Hint: $\ln(N!) \approx N \ln N - N$.)

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8) (Quantum Mechanics) A hydrogen atom is placed in a constant weak electric field of strength $\mathcal{E}$. Ignoring spin, what are the energies of the $n = 1$ and $n = 2$ levels including effects to first order in $\mathcal{E}$ (but ignoring second order effects) ?

You may want to use some of the following:

Radial wave functions $R_{n\ell}(r)$ (a is the Bohr radius):

$$R_{10}(r) = 2\, a^{-3/2}\, e^{-r/a}; \qquad R_{21}(r) = \frac{1}{2\sqrt{6}}\, a^{-3/2}\, \frac{r}{a}\, e^{-r/2a}$$

$$R_{20}(r) = \frac{1}{\sqrt{2}}\, a^{-3/2}\, (1 - \frac{r}{2a})\, e^{-r/2a}.$$

Spherical Harmonics $Y_\ell^m(\theta, \phi)$:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}; \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta; \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}.$$

An integral:

$$\int_0^\infty x^n e^{-x/a}\, dx = a^{n+1} n!$$

9) (Quantum Mechanics) The Hamiltonian for a system consisting of three distinguishable spin half particles is

$$H = A(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1).$$

where $\vec{S}_i$ is the spin of the i'th particle, and all the components of the spin of one particle commute with all the components of the spins of the others. What are the eigenvalues of H, and what are the degeneracies of each energy level?

10) (Quantum Mechanics) The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are the possible neutrino states produced and detected in experiments, but they are not necessarily eigenstates of the Hamiltonian. Rather, if the state is known to have momentum p, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

where

$$H|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\;|\nu_1\rangle$$
$$H|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\;|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some angle θ. If it is known that a neutrino was definitely a $|\nu_\mu\rangle$ when it was produced, what is the probability of detecting a $|\nu_e\rangle$ after it has traveled a distance L? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost (or even exactly) the speed of light, (so you can ignore corrections of the order $(1 - v/c)$ compared to terms of order 1) and state your result to first order in the difference $\Delta m^2 \equiv m_1^2 - m_2^2$.

(This is a simplified version of an actual neutrino oscillation experiment like the super-Kamiokande detector experiment last year. In reality there is a third neutrino $|\nu_\tau\rangle$.)

11) (Quantum Mechanics) The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Let $|\psi_n\rangle$, $n = 0, 1, 2 \ldots$, be the usual energy eigenstates.

a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0|\psi_0\rangle + c_1|\psi_1\rangle,$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that, not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}.$$

What is θ_1?

c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

12) (Quantum Mechanics) A beam of particles with uniform velocity v enters a region where some of them are absorbed. The absorption may be represented by introduction of a complex potential

$$V(\vec{r}) = V_R(\vec{r}) - iW_I(\vec{r}),$$

where both $V_R(\vec{r})$ and $W_I(\vec{r})$ are real. If the number of absorbing particles per unit volume is N, calculate the cross section per particle for the absorption.

13) (Quantum Mechanics) The wavefunction $\psi(t)$ of a stationary spin-$\frac{1}{2}$ particle in a magnetic field $\vec{B}(t)$ obeys the time-dependent Schrödinger equation

$$i\,\frac{\partial\psi(t)}{\partial t} = -\vec{\sigma}\cdot\vec{B}(t)\;\psi(t),$$

where $\vec{\sigma}$ is the vector containing the usual representation of the Pauli spin matrices. We have set $\hbar \equiv 1$, the Bohr magneton $\mu_B \equiv 1$ and the Landé g-factor (gyromagnetic ratio) $g_s \equiv 2$. Suppose that $\vec{B}(t) = (b\cos\omega t,\ b\sin\omega t,\ B_0)$, where b and B_0 are constants.

a) Show trivially that

$$i\frac{\partial\psi(t)}{\partial t} = -\begin{pmatrix} B_0 & be^{-i\omega t} \\ be^{i\omega t} & -B_0 \end{pmatrix}\psi(t).$$

b) By considering a solution of the form

$$\psi(t) = \begin{pmatrix} f_+(t)e^{iB_0t} \\ f_-(t)e^{-iB_0t} \end{pmatrix},$$

show that

$$i\frac{df_+}{dt}e^{i\Omega t} = -bf_-$$

and

$$i\frac{df_-}{dt} = -bf_+e^{i\Omega t},$$

where $\Omega = 2B_0 + \omega$.

c) Hence show that $f_-(t)$ satisfies the differential equation

$$\frac{d^2f_-}{dt^2} - i\Omega\frac{df_-}{dt} + b^2f_- = 0.$$

d) Given that $\psi(0) = \binom{1}{0}$ find f_- at subsequent times. Find the time when the particle is next in a pure spin-up state. Comment on the condition for which this time is the shortest.

14) (Stat. Mech./Thermo.) Some organic molecules have a spin-triplet ($S = 1$) excited state at energy $k_B\Delta$ above a spin singlet ($S = 0$) ground state. Here k_B is Boltzmann's constant, and Δ is a constant. You can express your answers also in terms of the Bohr magneton μ_B and the g-factor g (which you need not specify).

a) Find an expression for the magnetic moment in a magnetic field B, at temperature T.

b) Show that the susceptibility for $T \gg \Delta$ is given by $N(g\mu_B)^2/2k_BT$, where N is the total number of molecules in the system.

c) With the help of a diagram of energy levels versus field and a rough sketch of entropy versus field, explain how this system might be cooled by adiabatic magnetization (*not demagnetization*).