COMPREHENSIVE WRITTEN EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS
Thursday, March 30, and Friday, March 31, 2000

PART I - THURSDAY, MARCH 30

Important -- please read carefully.

The exam (8 hours) is in two parts:

Part 1 Electromagnetic Theory, Statistical Mechanics and Thermodynamics

March 30 7 Problems -- **DO ALL PROBLEMS.**

9:00-1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Part 2 Quantum Mechanics, Statistical Mechanics and Thermodynamics

March 31 7 Problems -- **DO ALL PROBLEMS.**

9:00-1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Instructions

1) This is a closed book exam and calculators are not to be used.

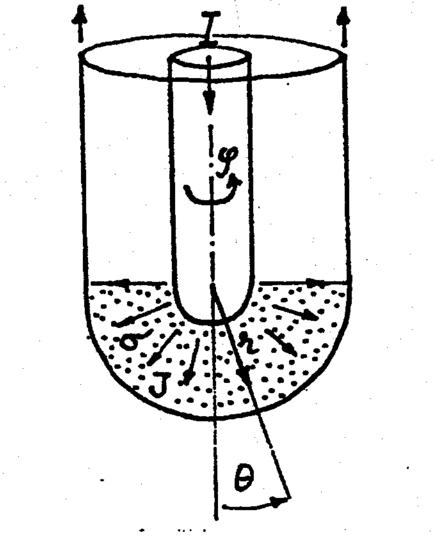
2) Work each problem on a separate sheet of paper. Use one side only.

3) Print your name and problem number on EACH **AND EVERY** page. (Note: Pages without names may not be counted.)

4) Return the problem page as the first page of your answer.

5) If a part of any question seems ambiguous to you, state clearly what your interpretation is and answer the question accordingly.

1. (E&M) Two concentric metallic electrodes have hemispherical ends. A resistive material fills the space between the concentric hemispheres. A current I flows through the inner electrode, through the resistive material and, finally returns through the outer electrode. The system is axially symmetric ($\partial/\partial\phi = 0$). Find the magnetic field inside the resistive material, $\boldsymbol{H}(\mathrm{r}, \theta)$.

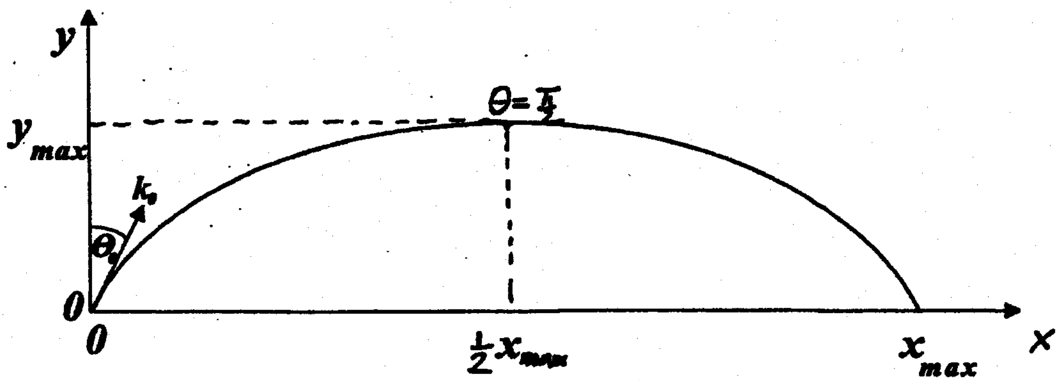


2. (E&M) Consider the refraction of a plane electromagnetic wave in a nonuniform medium where the refractive index varies with height y as $n(y) = 1 - y/L$ with L = const. At $y = 0$ the wave propagates at an angle $\theta_0 = 45^\circ$ with respect to the normal.

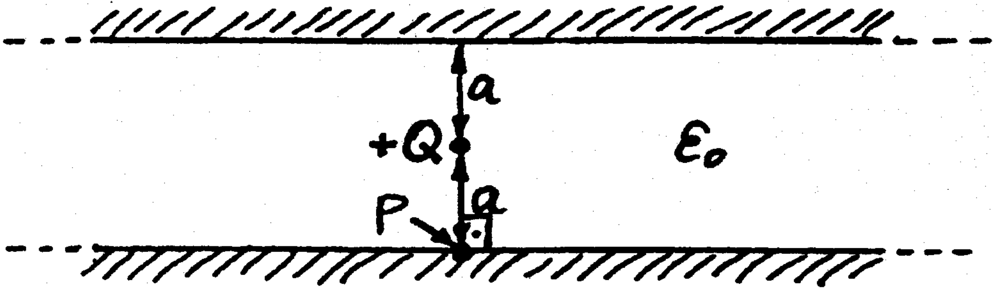
(1) Using Snell's law find the maximum height at which the wave propagates horizontally, $y_{max} = ?$

(2) Find the horizontal range x_{max} where the wave has refracted back to $y = 0$.

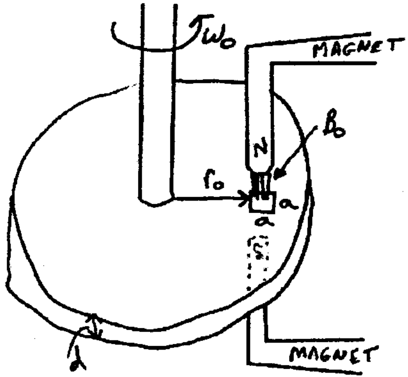
{Hint: Express the slope of the ray trajectory as a function of θ and solve $d\theta/dx$ using the integral expression $\int \frac{1}{\sin\theta} d\theta = \frac{1}{2} \ln \frac{1-\cos\theta}{1+\cos\theta}$ }



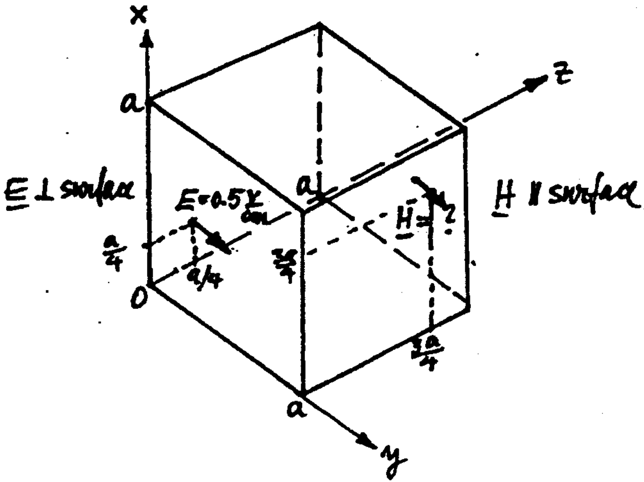
3. (E&M) A point charge +Q is located in the middle between two infinite, conducting planes separated by 2a as shown below. Find the approximate surface charge σ at point P.



4. (E&M) *Eddy current brake.* Calculate the torque tending to slow the rotation of the conducting disk shown below. The braking is generated by turning on a small region of uniform magnetic field B_0 over an area $a \times a$ at a distance r_0 from the axis of rotation of the disk, where $a \ll r_0$. The disk is rotating at an angular velocity of ω_0 and has a thickness d and a resistivity ρ. Find the torque in terms of the quantities given.



5. (E&M) A cavity resonator consists of an empty cube of sides a = 1 cm. What is the lowest resonance frequency? For the fundamental mode the electric field strength at x = a/4, y = 0, z = a/4 is found to be 0.5 V/cm. What is the direction and magnitude of the magnetic field at x = 3a/4, y = a, z = 3a/4?



6. (Stat. Mech./Thermo) Consider a heteronuclear diatomic molecule with moment of inertia, I. In this problem only the rotational motion of the molecule should be considered.

a) Using classical statistical mechanics, calculate the specific heat of the system at temperature T.

b) In quantum mechanics, this system has energy levels:

$$E_j = \frac{\hbar^2}{2I} j(j+1), j = 0,1,2\cdots$$

where the jth level is $(2j+1)$-fold degenerate. Using quantum statistical mechanics, find expressions for the partition function Z and the average energy $\langle E \rangle$ as a function of temperature T. Do not attempt to evaluate these expressions.

c) By simplifying your answer to (b), derive an expression for the heat capacity that is valid at very low temperatures. In what range of temperatures is your expression valid?

d) By simplifying your answer to (b), derive a high temperature approximation to the specific heat. What is the range of validity of your approximation?

7. (Stat. Mech./Thermo) Consider a three-dimensional system of electrons in a magnetic field. Ignore the interactions between electrons and the orbital effect of the field. The Hamiltonian is then:

$$H = \frac{-\hbar^2}{2m}\sum_i \nabla_i^2 - \frac{g\mu_B}{\hbar} B \sum_i S_i^z$$

where $\mu_B = e\hbar/2mc$.

a) The Fermi wavevectors for up- and down-spin electrons will be different for $B \neq 0$. What are $k_F^\uparrow$ and $k_F^\downarrow$ in terms of B and E_F?
(Recall $E_F = \mu\ (T=0)$, where μ is the chemical potential)

b) What is the relationship between E_F, n (the electron density), and B?

c) Find $S^z = \sum_i S_i^z$ in terms of n and B, to first-order in B.

d) Compute the zero-temperature Pauli spin susceptibility

$$\chi_{spin} = \left(\frac{\partial S^z}{\partial B}\right)$$

How does it compare to the corresponding classical result?

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PART II - FRIDAY, MARCH 31

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Part 1	Electromagnetic Theory, Statistical Mechanics and Thermodynamics
March 30	7 Problems -- **DO ALL PROBLEMS.**
9:00-1:00	This part will be collected at the end of four hours. Each problem counts for 20 points; the total is 140.
Part 2	Quantum Mechanics, Statistical Mechanics and Thermodynamics
March 31	7 Problems -- **DO ALL PROBLEMS.**
9:00-1:00	This part will be collected at the end of four hours. Each problem counts for 20 points; the total is 140.

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8. (Quantum Mechanics) Consider an electron moving in crossed electric and magnetic fields, $\boldsymbol{E} = (0, E, 0)$, $\boldsymbol{B} = (0, 0, B)$, where E and B are constants.

a) Write down Schrodinger's equation (in any gauge).

b) This equation is separable in the right gauge. Reduce it to an ordinary differential equation.

c) Find the energy eigenvalues.

9. (Quantum Mechanics) Consider the proton to be a spherical shell of charge with radius R. (Note: you may use $R \ll a_0$ throughout, where a_0 is the Bohr radius.)

(a) Using first order perturbation theory, calculate the change in the binding energy of hydrogen due to the non point-like nature of the proton.

(b) Does the sign of your result make physical sense? Explain.

(c) Estimate the magnitude of this correction, and express your answer in eV units.

10. (Quantum Mechanics) Consider the Hamiltonian of a one-dimensional harmonic oscillator $-(d/dx)^2 + x^2$. Take it to be known that the spectrum is discrete. Prove that the eigenvalues are positive. Then determine (no guessing!) the wave function and the energy of the ground state.

11. (Quantum Mechanics) A simple model for the dynamics of an atom near an impenetrable wall is obtained by considering a particle in the presence of the following one-dimensional potential

$$V(x) = \begin{cases} +\infty & x \leq 0 \\ -V_0\delta(x-d) & x \geq 0 \end{cases}$$

with $V_o > 0$ and $d > 0$.

(a) Obtain a simple transcendental equation for the energy of a bound state;

(b) Solve this equation (to leading non-trivial dependence in d) when the wall is "far away" from the atom, taking care of quantitatively defining "far away".

(c) What is the exact condition on V_0 and d for the existence of at least one bound state.

(d) Can the system ever have more than a single bound state ?

12. (Quantum Mechanics) Consider the Hamiltonian for a general top with principal moments of inertia I_1, I_2, I_3.

(a) For the symmetric top with $I = I_1 = I_2 \neq I_3$, derive all the energy levels.

(b) A slightly asymmetric top has $2\Delta = I_1 - I_2 \neq 0$, $2I = I_1 + I_2$ and $\Delta << I$, with $I \neq I_3$. Compute the $j = 0$ and $j = 1$ energies up to and including first order in Δ.

13. (Stat. Mech./Thermo) A system of 1 mole of ideal gas has an initial volume of V_1 and a temperature T_1,

(a) When the state of the system is changed to a volume of V_2 and a temperature of T_2, show that the entropy change is

$$\Delta S = c_v \ln\left(\frac{T_2}{T_1}\right) + R\ln\left(\frac{V_2}{V_1}\right)$$

(b) Assuming that the system undergoes the following process,
(1) from (p_1, V_1) to (p_2, V_2) by adiabatic expansion
(2) from (p_2, V_1) to (p_2, V_2) by constant-pressure compression
(3) from (p_2, V_1) to (p_1, V_1) by constant-volume heat absorption
show that the efficiency of this process is given by

$$\eta = 1 - \gamma \frac{\frac{V_2}{V_1} - 1}{\frac{p_1}{p_2} - 1}$$

where $\gamma \equiv c_p/c_v$, c_p is the specific heat at constant pressure and c_v is the specific heat at constant volume.

14) (Stat. Mech./Thermo.) An ideal monatomic gas of N particles, each of mass m, is in thermal equilibrium at absolute temperature T. The gas is contained in a cubical box of side L, whose top and bottom sides are parallel to the earth's surface. The effect of earth's uniform gravitational field on the particles should be considered, the acceleration due to gravity being g.

a) What is the average kinetic energy of a particle?

b) What is the average potential energy of a particle?