WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Thursday, March 29, and Friday, March 30, 2001

PART I - THURSDAY, MARCH 29

Important – please read carefully.

The exam (8 hours) is in two parts:

Part 1 Quantum Mechanics, Thermodynamics, Statistical Mechanics

March 29 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

PART 2 Electromagnetic Theory, Thermodynamics, Statistical Mechanics

March 30 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Instructions

1) This is a closed book exam and calculators are not be used.

2) Work each problem on a separate sheet of paper. Use one side only.

3) Print your name and problem number on EACH **AND EVERY** page. (Note: Pages without names may not be counted.)

4) Return the problem page as the first page of your answers.

5) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

1. *Quantum Mechanics.*

Consider a system of three spin-1/2 moments, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$. The permutation operator P_{12} exchanges spins 1 and 2:

$$P_{12}\ |\sigma_1, \sigma_2, \sigma_3\rangle = |\sigma_2, \sigma_1, \sigma_3\rangle$$

where $\sigma_{1,2,3} = \pm\frac{1}{2}$ are the eigenvalues of S_1^z, S_2^z, S_3^z. The permutation operator P_{123} performs a cyclic permutation on spins 1,2, and 3 so that $2 \to 1$, $1 \to 3$, $3 \to 2$.

$$P_{123}\ |\sigma_1, \sigma_2, \sigma_3\rangle = |\sigma_2, \sigma_3, \sigma_1\rangle$$

(a) Express P_{12} in terms of the spin operators $\mathbf{S}_1$, $\mathbf{S}_2$.
(b) Express P_{123} in terms of the spin operators $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$.

2. *Quantum Mechanics.*

An electron is injected into a region where there is a constant magnetic field of magnitude B. At $t = 0$, the direction of the electron's motion is perpendicular to the magnetic field, and it is completely polarized so that its spin is definitely along the direction of the beam.

Let Θ be the angle between the electron's momentum and the expectation value of its spin. At $t = 0$, $\Theta = 0$. What is Θ as a function of the time t? [Calculate the time-dependence of the momentum classically.] Express your answer in terms of the gyromagnetic ratio g of the electron. Leave g arbitrary – don't set it exactly equal to 2.

3. *Quantum Mechanics.*

A neutron (mass M) scatters off a very heavy nucleus, and the force between them is given by a Yukawa potential:

$$V(r) = V_o \frac{e^{-\mu r}}{\mu r}$$

(a) Imagine you could find the solution $\psi(\mathbf{r})$ to the time-independent Schrödinger equation (with an incident wave in the $+z$ direction) with this potential for positive energy E. Write a formula for the scattering amplitude in terms of this wave function. Don't try to calculate $\psi(\mathbf{r})$. Define any symbols you introduce, other than those in $V(r)$ above and natural constants.

(b) What is the first Born approximation to the scattering amplitude $f(\theta, \phi)$?

(c) What is the total cross section in the limit that the scattering neutron has zero kinetic energy?

4. *Quantum Mechanics.*

The simplest approximation for the Hamiltonian of an electron in a hydrogen atom is

$$H_o = \frac{\mathbf{p}^2}{2m} - \frac{\alpha\hbar c}{r}$$

where $\alpha \approx 1/137.036$ is a dimensionless constant. In cgs units, the electric charge e is related to α by $e^2 = \alpha\hbar c$.

(a) In this approximation, what are the energy levels of the hydrogen atom and what is the degeneracy of each level?

(b) There are some corrections to H_o that give rise to a small correction to the energy levels called the fine structure. What are the effects that give rise to the fine structure? Just describe them briefly – don't try to remember the formulas. What is the order of magnitude of the fine structure splitting compared to the splitting between the eigenvalues of H_o? Why?

(c) Consider the states in the first excited level with the approximation H_o above. Into how many levels are these states split, and what is the degeneracy of each level? What are the quantum numbers of the states in each level?

(d) There is a further splitting called the hyperfine structure. What is the effect that causes the hyperfine structure? Here too just describe it briefly. Into how many levels is the ground state level (i.e. all the states in the lowest energy level when hyperfine structure is ignored) split by the hyperfine effect, and what is the degeneracy of each level? Why is the hyperfine splitting small compared to the fine structure splitting?

5. *Quantum Mechanics.*

A particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave function.

6. *Statistical Mechnics and Thermodynamics*

Calculate the collision frequency for the collisions between the molecules of a gas and a fixed sphere of diameter D. The molecules have an average diameter d. The gas has a temperature T.

7. *Statistical Mechnics and Thermodynamics*

This is an essay question. Answer **two** of the following three questions.

(a) You are asked about the second law of thermodynamics, and you give one of the formulations, that there is no process the sole effect of which is the conversion of heat into work. The inquirer then points out that a steam engine converts heat into work. Explain how this is not a violation of the second law of thermodynamics. Your explanation should include an analysis of the steam engine, and a discussion of heat engines in general.

(b) You read an article in a physics journal in which a group of researchers announce that they have cooled a system to absolute zero. Discuss why one ought to be skeptical of this claim. Invoke the appropriate laws of thermodynamics.

(c) Explain, using the laws of thermodynamics, why a substance cannot have a negative heat capacity.

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8. *Electricity and Megnetism*

One hemisphere of a metallic sphere of radius R is kept at a potential V while the other hemisphere is kept at a potential of $-V$.

(a) What is the approximate potential and electric field a far away distance r from the center of the sphere. Keep only the leading contribution in R/r.

(b) Suppose the the potential V varies in time as $V_0 e^{-i\omega t}$ where $\omega R \ll c$. What is the electric field far away from the sphere? Again keep only the leading contribution in R/r. (If you can't figure out an exact expression then explain the generic behavior.)

9. *Electricity and Megnetism*

A charged particle moves in a plane perpendicular to a magnetic field $\vec{B}$, which is uniform in space but varies very slowly with time.

(a) Find a relation between the momentum p, the magnetic field B, and the instantaneous cyclotron (or gyration) radius a_0 of the particle's trajectory. (The radius will change very slowly in time as the B field varies.)

(b) Using Faraday's law, derive an approximate relation between the magnitude of the induced electromotive force around the orbit, the time derivative of B and the intantaneous radius a_0.

(c) Utilizing your answer to the previous parts, or otherwise, show that p^2/B remains constant in time.

10. *Electricity and Megnetism*

A π^0 of velocity v_0 decays in flight into two photons $\pi^0 \to 2\gamma$. Compute the minimum and maximum values of the energies of the produced photons as a functions of v_0.

11. *Electricity and Megnetism*

Consider the penetration of a magnetic field $\mathbf{B}$ into a conducting medium by diffusion and convection.

The medium obeys an Ohm's law of the form $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$, where $\mathbf{E}$ is the electric field, $\mathbf{J} = -ne\mathbf{v}$ is the current density of electrons of velocity $\mathbf{v}$ and number density n, and η is a constant uniform scalar resistivity.

(a) Using Faraday's law and Ampere's law (neglect displacement current) obtain a differential equation for the magnetic field.

(b) Now consider a simple boundary problem: the conducting medium is located in the half-space $x > 0$. There exists a density gradient of scale length $L = n/(dn/dy)$. At t=0 a uniform field B_0 along $\mathbf{z}$ is applied in the space $x < 0$. Write down the differential equation for the field $B_z(x,t)$. Identify which terms describe field diffusion ($t \propto x^2$) and convection ($t \propto x$).

(c) Show that the solution $B = [1 - (kB_0/D)x]^{-1}$ satisfies the differential equation in steady-state where $k = (\mu_0 neL)^{-1}$ and $D = \eta/\mu_0$.

(d) Show that in the absence of diffusion ($\eta = 0$) a propagating field $B_z(x - vt)$ satisfies the differential equation. Find the propagation velocity v in terms of B_0 and ∇n.

12. *Electricity and Megnetism*

A magnetic field is given by

$$\mathbf{B} = (B_x, B_y, B_z) = ((1+\gamma)x, (-1+\gamma)y, -2\gamma z)$$

where γ is a constant.

(a) Show that this field satisfies Maxwell's equations and may be derived from a scalar potential.

(b) For $\gamma = 0$, find the equation for field lines, the vector potential, and show that field lines are lines of constant vector potential.

(c) Sketch field lines for three parameter values $\gamma = 0$, $\gamma = 1/3$, $\gamma = 1$.

13. *Statistical Mechnics and Thermodynamics*

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\vec{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

14. *Statistical Mechnics and Thermodynamics*

(a) What is the free energy (as a function of temperature, T, volume, V, and particle number, N) of a ideal gas obeying Maxwell-Boltzmann statistics?

(b) Assume that the ideal gas is made up of hydrogen atoms. Now the free energy must include a contribution reflecting the different possible electronic excited states of the hydrogen atoms. Show that this contribution diverges. What cuts off this divergence in a real gas?