

WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S  
DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE  
DEPARTMENT OF PHYSICS

Thursday, September 19, and Friday, 20, 2002

**PART I – THURSDAY, September 19, 2002**

**Important – please read carefully.**

The exam (8 hours) is in two parts:

**PART 1**          Quantum Mechanics, Thermodynamics, Statistical Mechanics

**Sept 19**                  7 Problems – **DO ALL PROBLEMS.**

9:00–1:00          This part will be collected at the end of four hours.  
Each problem counts for 20 points; the total is 140.

**PART 2**          Electromagnetic Theory, Thermodynamics, Statistical Mechanics

**Sept 20**                  7 Problems – **DO ALL PROBLEMS.**

9:00–1:00          This part will be collected at the end of four hours.  
Each problem counts for 20 points; the total is 140.

**Instructions**

- 1) This is a closed book exam and calculators are not be used.
- 2) Work each problem on a separate sheet of paper. Use one side only.
- 3) Print your name and problem number on **EACH AND EVERY** page. (Note: Pages without names may not be counted.)
- 4) Return the problem page as the first page of your answers.
- 5) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

## 1. *Quantum Mechanics.*

A Stern-Gerlach apparatus is adjusted so that the  $z$ -component of the spin of an electron (spin- $1/2$ ) transmitted through it is  $-\hbar/2$ . A uniform magnetic field in the  $x$ -direction is then switched on at time  $t = 0$ .

- (a) What are the probabilities associated with finding the different allowed values of the  $z$ -component of the spin after time  $T$ ?
- (b) What are the probabilities associated with finding the different allowed values of the  $x$ -component of the spin after time  $T$ ?

## 2. Quantum Mechanics.

The Hamiltonian for a spinless charged particle in a magnetic field is

$$H = \frac{1}{2m} \left[ \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right]^2,$$

where the magnetic field  $\mathbf{B}$  is related to the vector potential  $\mathbf{A}$  by  $\mathbf{B} = \nabla \times \mathbf{A}$ . Here,  $e$  is the charge of the particle,  $m$  the mass,  $c$  the velocity of light and  $\mathbf{p} = (p_x, p_y, p_z)$  is the momentum of the particle. Let  $\mathbf{A} = -B_0 y \hat{x}$ , corresponding to the magnetic field  $\mathbf{B} = B_0 \hat{z}$ .

- (a) Find the energy levels of the particle.
- (b) Would the energy levels change if we chose  $\mathbf{A}$  to be  $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$ ? Give reasons for your answer.

### 3. Quantum Mechanics.

A charged particle of charge,  $q$ , and mass,  $m$ , is bound in a one-dimensional harmonic oscillator potential  $V = \frac{1}{2}m\omega^2x^2$ , where  $\omega$  is the frequency of the oscillator. The system is then placed in an electric field  $E$  that is constant in space and time.

- (a) Calculate the shift of the ground state energy to order  $E^2$ .
- (b) What are the third and the higher order (in  $E$ ) shifts in the ground state energy? Give reasons for your answer.

Hint: If  $n$  labels the eigenstates of the unperturbed harmonic oscillator, then  $\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} [\sqrt{n'}\delta_{n,n'-1} + \sqrt{n'+1}\delta_{n,n'+1}]$ .

#### 4. Quantum Mechanics.

Consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{Ze^2}{r} \quad (1)$$

- (a) What is the ground state energy of this Hamiltonian?
- (b) What is the expectation value of the potential energy  $\langle -\frac{Ze^2}{r} \rangle$  in the ground state?
- (c) What is the expectation value of the kinetic energy  $\langle -\frac{\hbar^2}{2m} \nabla^2 \rangle$  in the ground state?

## 5. Quantum Mechanics.

In the Born-Oppenheimer approximation, the electrons are treated quantum mechanically, while the atomic nuclei are treated classically. The electronic energy is calculated as a function of the spacing between the nuclei. The sum of the electronic energy and the potential energy due to nuclei-nuclei interactions is minimized. The nuclear kinetic energy is neglected. As a toy model, we will consider the formation of a diatomic molecule in one dimension. Let us suppose that the electron is at  $x$  and the nuclei are at  $X_1, X_2$ . We assume that the interaction between an electron and a nucleus is  $V(x - X_i) = -V_0 \delta(x - X_i)$  for  $i = 1, 2$ ,  $V_0 > 0$ . The interaction between nuclei is  $U(X_1 - X_2) = \frac{Z^2 e^2}{|X_1 - X_2|}$ .

- (a) Suppose that the nuclei are a distance  $a$  apart. What is the ground state energy of an electron to order  $a$  if  $\frac{mV_0 a}{\hbar^2} \ll 1$  ( $m$  is the electron mass).
- (b) Consider a diatomic molecule composed of an electron and two nuclei. Using the Born-Oppenheimer approximation, find the separation between the two nuclei if  $V_0 \gg Z^2 e^2$ . You need only compute the separation to lowest order in  $Ze/V_0^{1/2}$ .

## 6. *Statistical Mechanics and Thermodynamics*

A gas of  $N$  highly relativistic, and non-interacting, spin  $1/2$  Fermions occupies a volume  $V$  at a temperature that is effectively equal to zero.

- (a) Find the pressure on this gas.
- (b) Based on the calculation you have just done, show what (extreme) inequality must be satisfied in order that the assumption of a temperature that is “effectively equal to zero” is justified.
- (c) Suppose that energy of the system due to gravitational self-attraction goes as  $-\mathcal{A}N^2V^{-1/3}$ , where  $\mathcal{A}$  is a constant. What does this and your result for the pressure imply about the stability of this system, assuming that gravitational attraction is what keeps it together?

## 7. Statistical Mechanics and Thermodynamics

A chain consists of  $N$  links that can freely rotate in two dimensions. The links are joined end-to-end, as shown below.

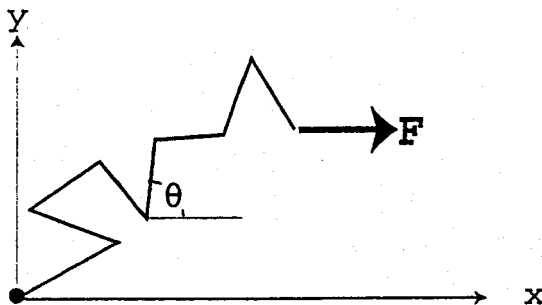


Figure 1: The freely-jointed chain. The angle,  $\theta$ , between one link and the  $x$ -axis is shown.

The chain is subjected to a tension,  $F$ , in the  $x$ -direction, as indicated. The tension is applied at the end of the chain, so that the total energy of the chain is given by

$$E = -Fl \sum_{i=1}^N \cos \theta_i$$

where  $\theta_i$  is the angle that the  $i^{\text{th}}$  link makes with the  $x$ -axis, and  $l$  is the length of each link in the chain.

- Calculate the partition function of this chain.
- From the partition function, find the relationship between the extension of the chain in the  $x$ -direction and the tension,  $F$ , assuming that the temperature is  $T$ .
- When the tension,  $F$ , is small, the extension-versus-tension expression implies a spring constant for the freely jointed chain. What is this effective spring constant?

If integrals do not evaluate to elementary functions in parts a and b, it is not necessary to attempt to reduce them. Leave them as integrals. However, in part c, it is necessary to come up with something explicit.

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**PART II – FRIDAY, September 20, 2002**

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- Sept 19      7 Problems – **DO ALL PROBLEMS.**
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**Instructions**

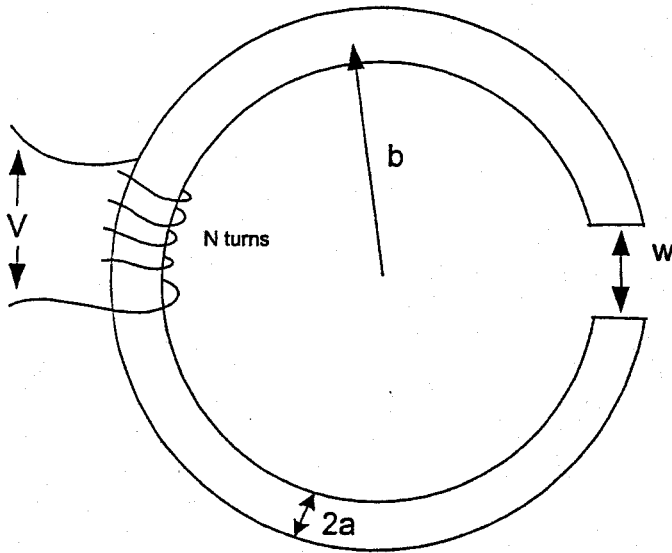
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Radiating charges

- (a) A point charge  $q$  under acceleration  $a(t)$  emits electromagnetic radiation. Give qualitative physical arguments why the radiated power,  $P$ , should be of the form  $P = Bq^2a^2$ , where  $B$  is a proportionality constant. Determine by dimensional analysis the dependence of  $B$  on fundamental physical constants. Explain how and why the exact expression for  $B$  differs from this estimate.
- (b) A point charge  $q$  has mass  $m$  and is attached to a spring (of spring constant  $\kappa$ ) hanging from a fixed support above an infinite horizontal **conducting** plane. The charge is set in motion with amplitude  $A < h$ , the equilibrium height of the charge above the conducting plane. Calculate its instantaneous radiating power.

## 9. Electricity and Magnetism

A D.C. electromagnet is constructed from a cylindrical soft-iron bar with radius  $a$ . The relative magnetic permeability of the iron is  $\mu$ . The bar is bent into a C-shape as shown below with radius  $b$ . The width of the small gap is  $w$ . The magnet is energized by winding a coil of copper wire  $N$  turns tightly around the bar and connecting the coil to a D.C. power supply with voltage  $V$ . The copper wire has a resistivity  $\rho$ , and radius  $r_{\text{wire}}$ . Assume  $r_{\text{wire}} \ll a \ll b$  and ignore fringe-field effects.



- What is the steady-state value of the magnetic field  $B$  in the gap?
- What is the time constant governing the response of the current in the coil when the voltage is turned on? (Assume  $\mu$  is constant.)

10. *Electricity and Magnetism*

A point charge  $q$  is inside a hollow, grounded, conducting sphere of inner radius  $a$ . Use the method of images to find

- (a) the potential inside the sphere;
- (b) the induced surface-charge density at the point on the sphere nearest to  $q$ ;
- (c) the magnitude and direction of the force acting on  $q$ .
- (d) Is there any change in the solution if the sphere is kept at a fixed potential  $V$ ? If the sphere has a total charge  $Q$  on its inner and outer surface?

## 11. *Electricity and Magnetism*

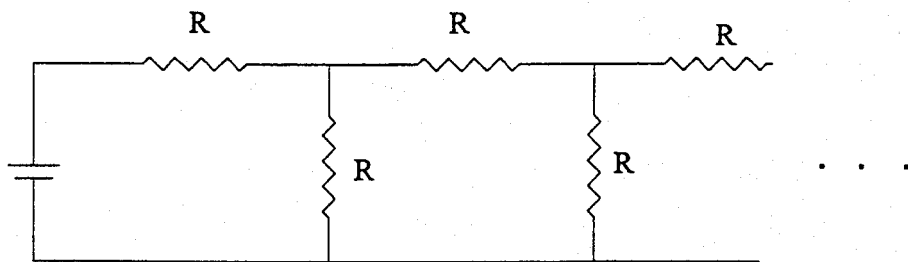
Describe how you would *measure* the following physical quantities:

- (a) An electrostatic field  $\mathbf{E}$ .
- (b) A vector potential  $\mathbf{A}$  defined by  $\mathbf{B} = \nabla \times \mathbf{A}$  in the gauge  $\nabla \cdot \mathbf{A} = 0$ .
- (c) The charge of an electron assuming its mass is known.
- (d) The speed of light or electromagnetic waves.
- (e) The electrical conductivity of a flame
- (f) The direction of wave propagation of a plane electromagnetic wave.

Please describe the approach and method as realistically as possible.

## 12. Electricity and Magnetism

A voltage is applied to the infinitely long resistor network shown below. Each resistor has the same resistance  $R$ . Calculate the power dissipated in each resistor.



### 13. *Statistical Mechanics and Thermodynamics*

Consider an ideal monoatomic gas in which each atom has two internal energy states, one an energy  $\Delta$  above the other. There are  $N$  atoms in a volume  $V$  at temperature  $T$ .

Find the a) chemical potential, b) free energy, c) entropy, d) pressure and e) heat capacity at constant pressure.

#### 14. Statistical Mechanics and Thermodynamics

In this problem, you will study the  $q$ -state Potts model using mean-field theory. The Hamiltonian is

$$H_{\text{Potts}} = -J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j} \quad (2)$$

where the 'spins'  $\sigma$  take values  $\sigma_i = 0, 1, 2, \dots, q-1$ . For  $q = 2$ , this is the Ising model.

(a) Show that  $H_{\text{Potts}}$  can be rewritten in the form

$$H_{\text{Potts}} = -\frac{J}{q} \sum_{\langle i,j \rangle} [(q-1)\mathbf{s}_i \cdot \mathbf{s}_j + 1] \quad (3)$$

where the vectors  $\mathbf{s}_i$  are constrained to take values in a set of  $q$  vectors in  $(q-1)$ -dimensional space,  $\{\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_q\}$  satisfying  $\mathbf{S}_a \cdot \mathbf{S}_b = 1$  if  $a = b$  and  $\mathbf{S}_a \cdot \mathbf{S}_b = -\frac{1}{q-1}$  if  $a \neq b$ .

(b) In mean-field theory, we approximate the Hamiltonian  $H_{\text{Potts}}$  by a mean-field Hamiltonian  $H_{\text{MF}}$ ,

$$H_{\text{MF}} = \sum_i \left[ \mathbf{h} \cdot \mathbf{s}_i - \frac{J}{q} \right] \quad (4)$$

in which there is no interaction between the different spins, but each spin is coupled to an effective magnetic field,  $\mathbf{h}$ . Calculate the partition function of  $H_{\text{MF}}$ .

(c) Using  $H_{\text{MF}}$ , compute  $\langle \mathbf{s} \rangle$ . Impose the self-consistency condition that the effective magnetic field is generated by the average spin  $\langle \mathbf{s} \rangle$  so that  $H_{\text{MF}}$  approximates  $H_{\text{Potts}}$ . Requiring self-consistency, derive (but do not solve) the mean-field equation for  $\langle \mathbf{s} \rangle$ . (For simplicity, you may assume that the tetrahedron is oriented so that one of the allowed values of  $\mathbf{s}_i$  is  $(0, 0, \dots, 0, 1)$ . Assume that  $\langle \mathbf{s} \rangle = s(0, 0, \dots, 0, 1)$  and find  $s$ .)