1. *Quantum Mechanics*

Two spin-half particles are in a state with total spin zero. Let $\hat{\boldsymbol{n}}_a$ and $\hat{\boldsymbol{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\boldsymbol{n}}_a$ and the spin of the second along $\hat{\boldsymbol{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\, \mathbf{s}_a \cdot \hat{\boldsymbol{n}}_a \mathbf{s}_b \cdot \hat{\boldsymbol{n}}_b \,|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

2. *Quantum Mechanics*

The van der Waals interaction between two neutral atoms is due to dipole-dipole interactions. Consider the following simplified $1D$ model. Each atom consists of a fixed nucleus of charge $+e$ and electron of charge $-e$, bound by a harmonic spring. Two such oscillators are a distance R ($\gg$ size of the atom) apart. The Hamiltonian of the system consists of the two harmonic oscillator terms plus a dipole-dipole perturbation.

(a) Write the perturbation part of the Hamiltonian

(b) Calculate the correction to the energy of the unperturbed ground state. This is the van der Waals interaction potential (hint: it should come out $\propto 1/R^6$.)

3. *Quantum Mechanics*

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron respectively. Normalize these states so that

$$\left\langle \mathbf{r}_+, \mathbf{r}_- \middle| \mathbf{r}'_+, \mathbf{r}'_- \right\rangle = \delta_3\left(\mathbf{r}'_+ - \mathbf{r}_+\right)\delta_3\left(\mathbf{r}'_- - \mathbf{r}_-\right)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \, \psi \rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interaction between the two particles.

(c) Let

$$\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$$

and

$$\mathbf{R} = \frac{\mathbf{r}_+ + \mathbf{r}_-}{2}$$

Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium*. For states with zero total momentum, write a formula for the possible negative values of the energy[1]? What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C|\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

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[1]Write your answer in terms of m, e^2 or α, $\hbar$, c, the Bohr radius, etc. You may use units in which $\hbar = c = 1$.

4. *Quantum Mechanics*

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L} + \mathbf{s}$. Conventionally the states are labeled $|n, l, j, m\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

(a) If the electron is in the state $|n, l, j, m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m?

(c) Let $\mathbf{J}_\pm = J_x \pm J_y$. What are

$$i)\left\langle 3,1,\frac{3}{2},\frac{3}{2}\right| J_+ \left|3,1,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$$

$$ii)\left\langle 3,1,\frac{3}{2},\frac{3}{2}\right| J_+ \left|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$$

$$iii)\left\langle 2,1,\frac{3}{2},\frac{3}{2}\right| p_z \left|2,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$$

$$iv)\left\langle 2,1,\frac{1}{2},-\frac{1}{2}\right| \mathbf{L}^2 \left|2,1,\frac{1}{2},-\frac{1}{2}\right\rangle = ?$$

$$v)\left\langle 3,2,\frac{3}{2},-\frac{1}{2}\right| \mathbf{J}^2 \left|3,2,\frac{3}{2},-\frac{1}{2}\right\rangle = ?$$

$$vi)\left\langle 3,1,\frac{3}{2},\frac{3}{2}\right| \mathbf{J}_z \left|3,1,\frac{3}{2},\frac{1}{2}\right\rangle = ?$$

(d) What is

$$\left\langle 1,0,\frac{1}{2},\frac{1}{2}\right| p_i p_j \left|1,0,\frac{1}{2},\frac{1}{2}\right\rangle = ?$$

(e) For given n, l, j, and m, what are the conditions on n', l', j', and m' so that

$$\langle n', l', j', m'|\, \mathbf{s}\cdot\mathbf{r}\, |n, l, j, m\rangle \neq 0\ ?$$

5. *Quantum Mechanics*

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2 \ldots$, be the usual energy eigenstates.

Part a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_o|\psi_o\rangle + c_1|\psi_1\rangle\,.$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_o|$ and $|c_1|$?

Part b) Choose c_o to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|\, x\, |\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

Part c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

6. *Statistical Mechanics*

If the specific heat of a gas of non-interacting fermions in d-dimensions varies with temperature as $C \sim T^{\alpha}$ for $k_B T \ll E_F$, then what is α? What is α for a system of non-interacting bosons?

7. *Statistical Mechanics*

Some organic molecules have a triplet excited state at energy $k_B\Delta$ above a singlet ground state.

(a) Find an expression for the magnetic moment in a field B in terms of Δ, B, the temperature T, the Bohr magneton μ_B, and the gyromagnetic ratio g.

(b) Show that the susceptibility for $T \gg \Delta$ is given by $N(g\mu_B)^2/2k_BT$, where N is the total number of molecules in the system.

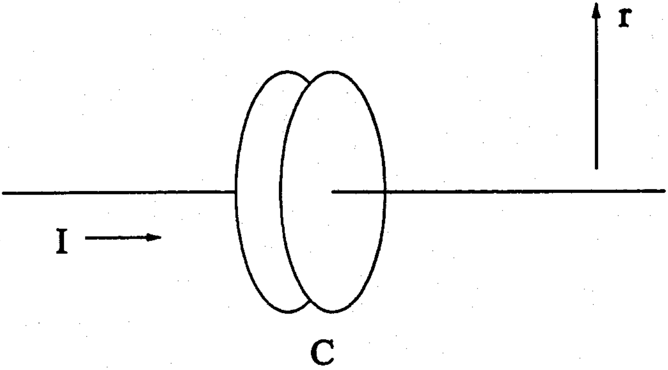
(c) With the help of a diagram of energy levels versus field and a rough sketch of entropy versus field, explain how this system might be cooled by adiabatic magnetization (*not demagnatization*).

8. *Electricity and Magnetism*

Consider a sphere of radius a with uniform magnetization $\mathbf{M}$, pointing in the z-direction. What are the magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ inside the sphere?

9. *Electricity and Magnetism*

A wire carrying current I is connected to a circular capacitor of capacitance C, as depicted in the figure. What is the magnetic field outside the wire, far from the capacitor (as a function of the distance r from the wire)? Using Maxwell's equations, explain why there is a magnetic field outside the capacitor. What is this magnetic field?



10. *Electricity and Magnetism*

The upper half-space is filled with a material permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\,\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

11. *Electricity and Magnetism*

Using general principles, find the radiated power in vacuum of a non-relativistic point charge q whose position is $\mathbf{r}(t)$. You do not need to find dimensionless proportionality constants (i.e. only find the dependence on q, $\mathbf{r}(t)$, and universal constants).

12. *Electricity and Magnetism*

(a) Show that the annihilation of an electron and a positron can produce a single massive particle (say, X), but not a single photon.

(b) A positron beam of energy E can be made to annihilate against electrons by hitting electrons at rest in a fixed-target machine or by hitting electrons moving in the opposite direction with the same energy E in an electron-positron collider (colliding-beam accelerator). Show that the minimum energy E_{min} of a positron beam needed to produce neutral particles X of mass $M \gg m_e$ (where m_e is the electron rest mass) is much greater in a fixed-target machine than in a collider.

13. *Statistical Mechanics*

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2} r M^2 + uM^4 - hM \tag{1}$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly-dependent on T and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α

(c) At $T = T_c$, $M \sim h^\delta$. What is δ?

14. *Statistical Mechanics*

Consider black body radiation at temperature T. What is the average energy per photon in units of kT?

You may find the following formulae useful:

$$\int_0^\infty \frac{x^3\,dx}{e^x - 1} = \frac{\pi^4}{15} \approx 2.4\,; \quad \int_0^\infty \frac{x^2\,dx}{e^x - 1} \approx 2.4$$