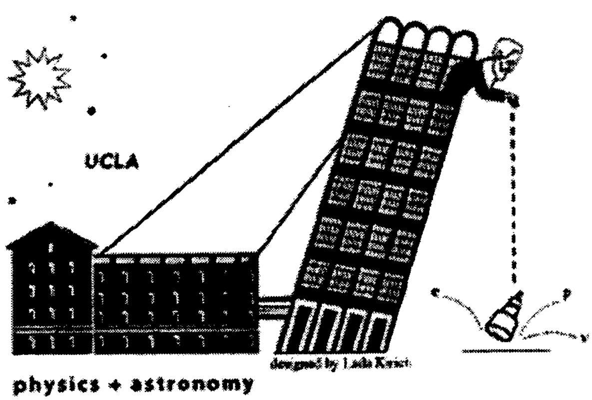
Fall 2006

Physics Comprehensive Exam

September 21, 2006 (Day 1)

- Please PRINT your name on all pages (including additional answer sheets)
- Use only paper provided to you
- Clip all pages together including problem page when submitting your completed exam



Quantum Mechanics Problem #1

a) For a spherically symmetric potential, show that the radial part of a wave function obeys the radial Schrödinger equation

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{l(l+1)}{2mr^2}+V(r)-E\right)u_l(r)=0$$

Assume that the potential $V(r)$ vanishes rapidly for large r and is less singular than $1/r^2$ for small r.

b) Derive the behavior of $u_l(r)$ for $r \to 0$.

c) Derive the behavior of $u_l(r)$ for large $r(r \to \infty)$ when it describes a bound state.

Quantum Mechanics Problem #2

The spin degree of freedom of a spin 1/2 particle with mass m can be described in a basis $|\pm\rangle$, where

$$\sigma_3 \,|+\rangle = +\,|+\rangle, \quad \sigma_3 \,|-\rangle = -\,|-\rangle,$$

and where σ_3 is the third Pauli matrix. The spin operator for a single fermion is $S_3 = \frac{\hbar}{2}\sigma_3$

a) Two identical fermions of spin 1/2 are initially assumed to be noninteracting. For this part of the problem take only the spin degrees of freedom into account. Construct the singlet state, i.e. the state for which the total spin of the two fermion system satisfies $S_3 = 0$, and $\mathbf{S}^2 = 0$.

Now consider that the two spin 1/2 fermions are both moving in a one dimensional infinite square well with potential

$$V(x) = \begin{cases} \infty & x < 0 \\ -a & 0 < x < L \\ \infty & x > L \end{cases}$$

For the rest of the problem take both the spin degrees of freedom and the spatial wavefunction into account.

b) What does the Fermi exclusion principle imply for the wavefunction of the two fermion system? What does this imply for the spatial wavefunctions of the singlet state?

c) Find the normalized wavefunction of the two fermion system which has the lowest energy and is a singlet. Find the energy eigenvalue for this state.

d) Now assume that there is a small interaction of the form

$$V_{int}(x_1, x_2) = -\alpha\, \delta(x_1 - x_2)$$

To lowest order in perturbation theory find the change in energy of the ground state due to the interaction.

Consider two flavours of massive neutrinos, denote $|\,\nu_e\rangle$ the electron neutrino flavour eigenstate and $|\,\nu_\mu\rangle$ the muon neutrino flavour eigenstate. These are related to the energy eigenstates $|\,\nu_1\rangle$ and $|\,\nu_2\rangle$ by

$$\begin{aligned} |\,\nu_\epsilon\rangle &= \cos(\theta)\,|\,\nu_1\rangle - \sin(\theta)\,|\,\nu_2\rangle \\ |\,\nu_\mu\rangle &= \sin(\theta)\,|\,\nu_1\rangle + \cos(\theta)\,|\,\nu_2\rangle \end{aligned}$$

a) Show that flavor eigenstates and energy eigenstates are related by a unitary transformation.

b) The energy of the eigenstate $|\,\nu_i\rangle$ is

$$E_i = \sqrt{\vec{p}^2 c^2 + m_i^2 c^4}, \quad i = 1, 2$$

Assume that an electron neutrino is produced in the sun with momentum $\vec{p}$ such that $|\vec{p}| >> m_i c$. Find the probability for the electron neutrino to oscillate into a muon neutrino after travelling a distance L.

Consider a quantum mechanical system with Hamiltonian

$$H = a^\dagger a$$

Where a and $a^\dagger$ are operators satisfying the following relations

$$a^2 = 0, \quad (a^\dagger)^2 = 0, \quad a^\dagger\, a + a\, a^\dagger = 1$$

a) Show that the Hamiltonian satisfies

$$H^2 = H$$

b) Find the eigenvalues of the Hamiltonian H

c) If $\mid 0\rangle$ is the **unique** normalized ground state of the system (i.e. the state with the lowest energy eigenvalue) find

$$a \mid 0\rangle = ?$$

Under the assumption above, what dimension can the complete Hilbert space of states have?

Quantum Mechanics Problem #5

A neutron (mass M) scatters off a very heavy nucleus, and the force between them is given by a Yukawa potential:

$$V(r) = V_o \frac{e^{-\mu r}}{\mu r}$$

a) Imagine you could find the solution $\psi(\vec{r})$ to the time-independent Schrodinger equation (with an incident wave in the $+z$ direction) with this potential for positive energy E. Write a formula for the scattering amplitude in terms of this wave function. Don't try to calculate $\psi(\vec{r})$. Define any symbols you introduce, other than those in $V(r)$ above and natural constants.

b) What is the first Born approximation to the scattering amplitude $f(\theta, \phi)$?

c) What is the total cross section in the limit that the scattering neutron has zero kinetic energy?

Name ____________________

Statistical Mechanics Problem #6

FQ06 - Sept. 21/22, 2006

Consider a set of spin one-half particles in a magnetic field, $\vec{H}$, oriented in the z direction. There is no interaction between the particles, and the energy of interaction of particles with this magnetic field is given by

$$E_H = -\mu_B \sum_{i=1}^{N} s_i H$$

where μ_B is proportional to a Bohr magneton, and the quantities s_i can take on the values ± 1. We will forget about the g factor for the moment. The Boltzmann factor governing the equilibrium statistics of these moments is $e^{-\beta E_H}$, where $\beta = 1/k_B T$.

a) obtain an expression for the total magnetic moment of this system of moments in the presence of the external magnetic field.

b) The specific heat of this system in the absence of an externally applied field is given by

$$C_H|_{H=0} = AT^2$$

Use this information and the solution to part a) of the question to obtain an expression for the magnetic Gibbs potential, $G(T, H, N)$ of this system of spins. This expression may contain undetermined coefficients.

c) From your result above, find the condition that applies when an external magnetic field is introduced or removed *adiabatically*.

Name ____________________

Statistical Mechanics Problem #7

FQ06 - Sept. 21/22, 2006

Consider a gas of noninteracting particles for which the kinetic energy of each has the following dependence on momentum

$$E(\vec{p}) = |\vec{p}|c$$

These particles obey Boltzmann statistics. There are N of them, occupying a volume V.

a) Find the partition function of this system of particles.

b) What is the heat capacity at constant pressure of this system of particles?

Fall 2006

Physics Comprehensive Exam

September 22, 2006 (Day 2)

- Please PRINT your name on all pages (including additional answer sheets)

- Use only paper provided to you

- Clip all pages together including problem page when submitting your completed exam

(a) Consider a grand canonical ensemble of particles, at fixed temperature T and in a container of volume V. Show that the mean square fluctuation in the number of particles $\overline{(\Delta N)^2}$ is:

$$\overline{(\Delta N)^2} = k_B T \frac{\partial \overline{N}}{\partial \mu}.$$

(b) Using the relation:

$$SdT - Vdp + Nd\mu = 0$$

express the solution in terms of $\left(\frac{\partial \rho}{\partial p}\right)_{T,V}$ where p = pressure and $\rho = \frac{N}{V}$ is the density of the system.

(c) Since intensive quantities are independent of extensive quantities by definition, we can change external constraints to obtain:

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,V} = \left(\frac{\partial \rho}{\partial p}\right)_{T,N}.$$

Using this relation, find an expression for $\overline{(\Delta N)^2}$ in terms of the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N}$.

Statistical Mechanics Problem #9 FQ06 - Sept. 21/22, 2006

Consider an idealized "white dwarf" star made up of ionized helium only. We make several simplifying assumptions, namely:

- there is no radiation pressure
- the electrons form a completely degenerate (i.e. "T = 0"), ultrarelativistic ("$m_e = 0$") Fermi gas
- the density ρ is uniform.

Set up the condition for mechanical equilibrium of the star under the opposing influences of the gravitational force and the pressure of the Fermi gas. You will find that, with these approximations, equilibrium is possible for only one particular value of the mass of the star, M (this mass is called the "Chandrasekhar limit"). Give the value of M in terms of fundamental constants.

[Note: this calculation, with $m_e = 0$, overestimates the pressure; in reality equilibrium is possible for masses smaller and up to the Chandrasekhar limit).

Name ____________________

Electromagnetics Problem #10

FQ06 - **Sept. 21/22, 2006**

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \to A$ as $r \to 0$ and $f(r) \to B$ as $r \to \infty$. $f(r)$ is a non-singular function.

a) What is the total charge of this system ? Give the answer in terms of $A, B, f(r)$ and (possibly) derivatives of $f(r)$.

b) Identify any point charges in this system and and give their location and charge.

c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of $A, B, f(r)$ and (possibly) derivatives of $f(r)$.

Electromagnetics Problem #11

FQ06 – Sept. 21/22, 2006

Consider an electromagnetic wave incident from vacuum onto a dielectric with a dielectric constant, ϵ. The surface normal lies along the $\hat{z}$ axis.

a) Derive the reflection and transmission coefficients if the wave is incident along the $\hat{z}$ direction.

b) Derive the reflection and transmission coefficients if the wave is incident in the $\hat{x}\hat{z}$ plane with an angle θ_i with respect to the surface normal and the electric field is in the y direction. Is there an angle for which there is no reflected energy?

Electromagnetics Problem #12

Consider an single electron interacting with electric and magnetic fields obtained from the corresponding scalar and vector potentials.

a) If the fields do not depend explicitly on time then the energy is conserved. Start from the equation for the time rate of change of energy for a single charged particle and derive the relativistically correct expression for the energy?

b) Consider a one-dimensional problem where the fields only depend on one spatial variable. Suppose the fields are described by a scalar potential of the form $\phi = \phi_0 cos(kz - \omega t)$. What is the constant of the motion in the laboratory frame now? Hint: Take a linear combination of the conservation of energy equation and the conservation of momentum equation. Use this constant to determine how large ϕ_0 must be in order that an electron that starts from rest at z=0 at t=0 is trapped by the wave and to determine the maximum energy that the electron can obtain.

c) Consider a fully three-dimensional case. If both the scalar and vector potential are functions of $(x, y, z - v_\phi t)$ where v_ϕ is the phase velocity, then the energy is no longer a constant. What is the new constant? Hint: Take a linear combination of the conservation of energy equation and the component of the conservation of momentum equation in the $\hat{z}$ direction.

Electromagnetics Problem #13

FQ06 - Sept. 21/22, 2006

- Consider a charge q moving on a circle of radius a (centered at the origin) on the x-y plane, with constant angular velocity ω.

a) In the dipole approximation, calculate the power radiated per unit solid angle in the direction defined by the azimuthal angle θ (i.e. θ is the angle with the z axis).

b) Still in the dipole approximation, what is the state of polarization of the radiation emitted in the direction θ = 0 ? And θ = π/2 ?

c) Going beyond the dipole approximation, show that radiation is emitted also at frequencies other than ω (what frequencies ?). You may want to follow the steps below:

- show that if ρ(**x**, t) is periodic in time (but not necessarily of the form $\rho(\mathbf{x})e^{-i\omega t}$) with period T = 2π/ω, then one can write:

$$\rho(x,t) = \frac{1}{2}\rho_0(x) + \sum_{n=1}^{\alpha} \mathrm{Re}\left[\rho_n(x)e^{-in\omega t}\right] \text{where } \rho_n(x) = \frac{2}{T}\int_0^T dt \rho(x,t)e^{in\omega t} \quad (n \geq 1)$$

- recall that the mutipole moments are:

$$q_{lm} = \int d^3x Y_{lm}(\theta,\varphi) r^l \rho(x)$$

Write ρ(**x**) in spherical coordinates for this problem, and using the expression above, find the frequencies at which the different multipole terms radiate.

Consider a rotating sphere with radius R. A charge Q is distributed homogeneously over the sphere. The sphere rotates counter clockwise around the z-axis with angular velocity ω. [See figure below].

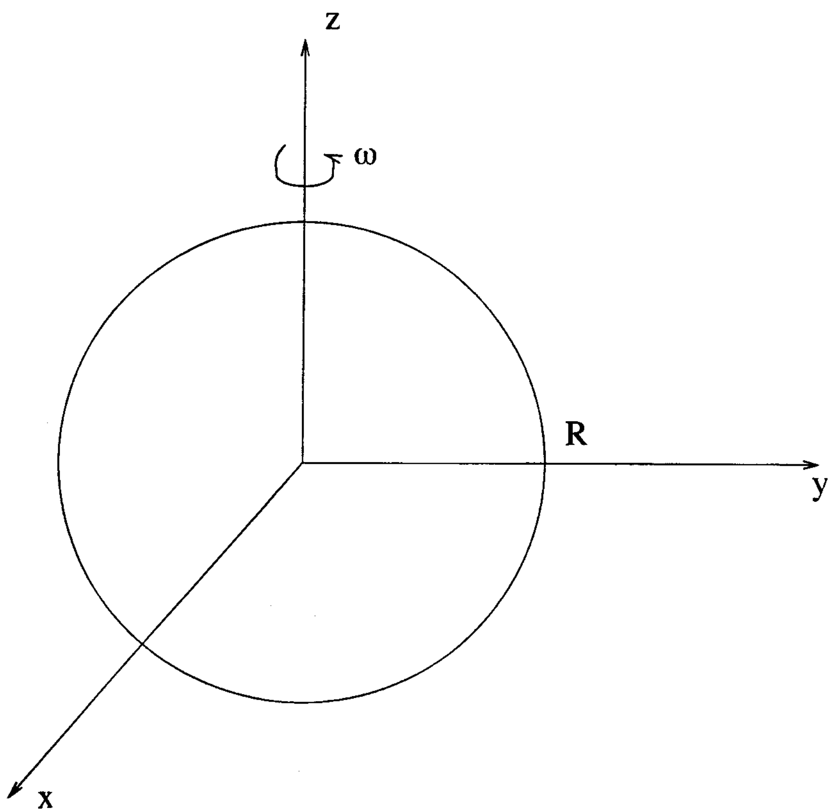


Figure for problem EM5

a) Find the charge density ρ and the current density $\vec{j}$ in terms of delta functions. Show that $\vec{\nabla} \cdot \vec{j} = 0$.

b) Find the vector potential $\vec{A}(\vec{x})$ in the Coulomb gauge ($\vec{\nabla} \cdot \vec{A} = 0$). Hint: To do the integral it is advantageous to choose $\vec{r} = r\vec{e}_z$ and choose $\vec{\omega}$ to be arbitrary.

c) Calculate the magnetic field $\vec{B}$ from the vector potential $\vec{A}$.