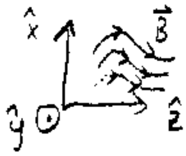
E & M study sheet

Examples taken from Morales notes (2004-05) and HW assignments:

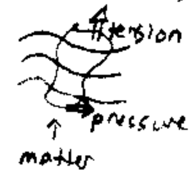
1) pressure & tension effects:



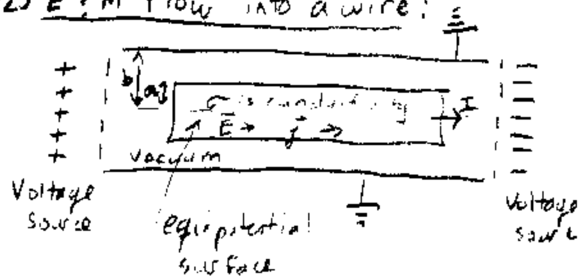
$\vec{B}(z) = B_0\,\hat{z} + \delta B\cos(kz)\,\hat{x}$, $B_0,\ \delta B =$ constants

→ since field lines are bent, matter must be present!

$$\vec{F} = m\vec{a} \Rightarrow \rho_m \frac{d}{dt}\vec{v} = \frac{1}{c}\vec{J}\times\vec{B} \quad \overset{\text{static condition}}{\&} \quad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} \Rightarrow \rho_m \frac{d}{dt}\vec{v} = \frac{1}{4\pi}(\nabla\times\vec{B})\times\vec{B} = \underbrace{\frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B}}_{\text{tension: variation along field lines matter}} - \underbrace{\nabla\left(\frac{B^2}{8\pi}\right)}_{\text{pressure}}$$



2) E & M flow into a wire:



$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$, assume steady-state $\frac{\partial}{\partial t}\to 0$

$\Rightarrow \nabla\cdot\vec{J} = 0$, $\vec{J} = \sigma\vec{E}$

$\Rightarrow \nabla\cdot\vec{E} = 0$ inside wire

So, $\vec{E}$ is uniform and pointing from + to −

inside wire: E is constant

$$|\vec{J}| = \sigma E \Rightarrow E = \frac{J}{\sigma} = \frac{I}{\sigma\pi a^2} \qquad r<a$$

$$\text{Since } \vec{E} = -\hat{z}\frac{\partial}{\partial z}\phi \Rightarrow \phi(r,z) = \frac{-I}{\pi a^2\sigma}z \qquad r<a$$

$$\text{For } a<r<b:\quad \nabla^2\phi = -4\pi\rho = 0 \overset{\text{in cylindrical coord}}{\Longrightarrow} \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\phi = 0$$

Now, $\phi(r,z) = \underset{\substack{\uparrow\\ \text{due to voltage source}}}{z}\,f(r) \Rightarrow \nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) = 0 \Rightarrow f(r) = k\ln r$

get k from B.C. $\phi(r=b,z) = 0 \Rightarrow f \to k'\ln(r/b)$

$$\Rightarrow \boxed{\phi = \frac{-Iz}{\pi a^2\sigma}\,\frac{\ln(r/b)}{\ln(a/b)}} \qquad \vec{E} = -\nabla\phi = \frac{I}{\pi a^2\sigma\ln(b/a)}\left[\ln\left(\frac{r}{b}\right)\hat{z} + \frac{z}{r}\hat{r}\right] \qquad \vec{S} = (\vec{E}\times\vec{B})\frac{c}{4\pi}$$

satisfies $\nabla^2\phi = 0$ & B.C $\qquad \vec{B} = \frac{2}{c}\frac{I}{r}\hat{\phi}$ (from Ampere's law)

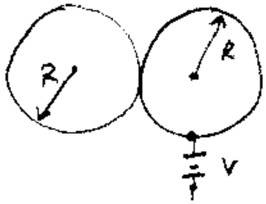
3) Toroidal field confinement

system has ϕ angle symmetry, $P_\phi = \text{constant}$

implies orbits are constrained since $P_\phi \propto \vec{v} + \vec{A}$

if you break symmetry, particle no longer confined

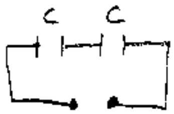
4)



2 grounded spheres

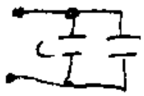
what is capacitance? if isolated, $C = \frac{q}{V}$, $V = \frac{q}{R}$ $\Rightarrow$ $C = R$

in series



$C_T = \frac{CC}{C+C} = \frac{C}{2} \Rightarrow C_T = \frac{R}{2}$

in parallel



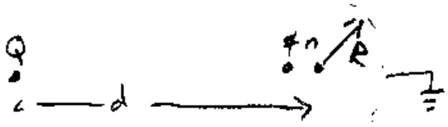
$C_T = C + C = 2C \Rightarrow C_T = 2R$

$\rightarrow$ the picture above looks like series so guess is $C_T = \frac{R}{2}$

$\rightarrow$ if you connect battery to one, they have same potential so looks like parallel, so guess $C_T = 2$

actual answer is $(2\ln 2)R$ $\Rightarrow$ not as good as $\parallel$ but better than series.

redraw picture



infinite # of image charges

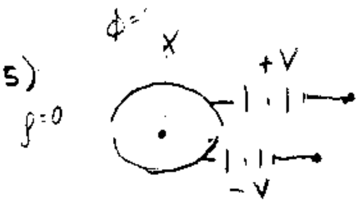
$q_n = \frac{Q}{n}(-1)^{n+1}$

$$q_{total} = 2\sum_{n=1}^{\infty} q_n = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}Q}{n} = 2Q\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \quad ; \; Q = VR$$

$$\text{note } \ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n} \Rightarrow \ln(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

$$q_{total} = 2VR\ln(2) \quad \therefore \; C = \frac{q_{total}}{V} = 2\ln(2)\,R$$

5)



what is dipole moment of this object

1st Find Φ

2nd Get $\vec{E} = -\nabla\Phi$

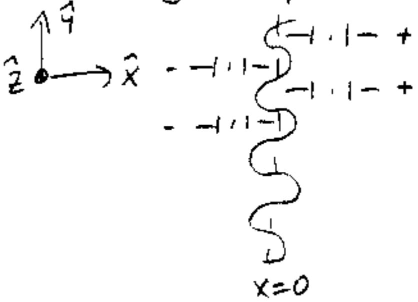
3rd evaluate $\Delta E_\perp = 4\pi\sigma$

4th Integrate $\int d^3r\, \vec{r}\, \rho(\vec{r})$

③

$$\Phi = \frac{\vec{p}\cdot\vec{r}}{r^2} = \frac{|\vec{P}|}{z^2} \quad \Rightarrow \quad |\vec{P}| = \frac{3}{2}a^2 V$$

6) Oscillating charge sheet



$\cdot\ \Phi = ?$ $\quad \to \infty$

$\Phi(x,y,z) = \Phi(x,y)$

Use separation of variables : $\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2 \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2 \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2 = \beta^2 + \alpha^2$

For our problem, $\beta \to K$, $\gamma \to 0$ (no variation) $\Rightarrow \gamma^2 = 0 = \alpha^2 + \beta^2 \Rightarrow \alpha^2 = -\beta^2 = -k^2$

$\therefore \boxed{\alpha = -iK}$

$x>0$: $\Phi(x,y) = [C_1\cos(Ky) + C_2\sin(Ky)][C_3 e^{-Kx} + C_4 e^{+Kx}]$

since we have $\Phi \to 0$ as $x \to \infty \Rightarrow C_4 = 0$

$$\Phi(x=0,y) = C_1\cos(Ky) + C_2\sin(ky) = V\cos(ky) \quad \Rightarrow C_2 = 0$$

So, $$\Phi(x,y) = \begin{cases} V\cos(Ky)\,e^{-Kx} & x>0 \\ V\cos(Ky)\,e^{Kx} & x<0 \end{cases}$$

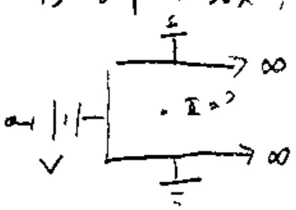
the general solution is $\Phi(x,y) = V\cos(Ky)\,e^{-|Kx|}$

From $\Delta E_\perp(x=0,y) = 4\pi\sigma(x=0,y) \Rightarrow \left(-\frac{\partial}{\partial x}\Phi\right)_{x=0^+} - \left(-\frac{\partial}{\partial x}\Phi\right)_{x=0^-} = 4\pi\sigma$

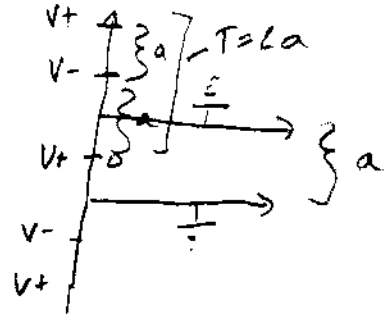
$\Rightarrow KV\cos(Ky) + KV\cos(Ky) = 4\pi\sigma$

$$\therefore \boxed{\sigma(x=0,y) = \frac{V\cos(Ky)}{2\pi}}$$

7) 3-plate box, infinite in $\hat{z}$ direction



$\Rightarrow$

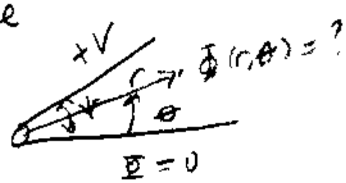


$$\Phi(x,y) = \sum_{m=0}^{\infty} b_m \cos(k_m y)\, e^{-k_m x} \overset{B.C.}{\Rightarrow} k_m = \frac{(2n+1)\pi}{a}$$

From orthogonality (at $x=0$) $\Rightarrow$ $b_m = \frac{4V}{k_n a}(-1)^n$

$$\therefore \Phi(x,y) = \sum_{n=0}^{\infty} \frac{4V(-1)^n}{(2n+1)\pi} \cos\left[\frac{(2n+1)\pi}{a} y\right] e^{-\frac{(2n+1)\pi x}{a}}$$

8) wedge



$$\nabla^2 \Phi = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\Phi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Phi = 0$$

if θ is not restricted:

$$\Phi(r,\theta) = (a r^{\alpha} + b r^{-\alpha})(c \cos\alpha\theta + d \sin\alpha\theta)$$

if θ is restricted

$$\Phi(r,\theta) = (a_0 + b_0 \ln r)(c_0 + d_0 \theta)$$

For our case, θ is restricted, so

$$\Phi(r,\theta) = (a_0 + b_0 \ln r)(c_0 + d_0 \theta)$$

Apply B.C.:

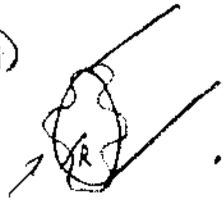
$\Phi(r, \theta = 0) = 0 \;\; \forall r \Rightarrow b_0 = 0$; $\Phi(r, \theta = \psi) = V \Rightarrow c_0 = 0$

So, $a_0 d_0 \psi = V \Rightarrow a_0 d_0 = \frac{V}{\psi}$

Then, $\boxed{\Phi(r,\theta) = \frac{\theta}{\psi} V}$ $\Rightarrow \vec{E} = -\nabla \Phi = -\hat{r}\frac{\partial}{\partial r}\Phi - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\Phi$

$\therefore \boxed{\vec{E} = \frac{-V}{\psi r}\hat{\theta}}$

9)



$\cdot\ \Phi(r,\theta) = ?$
$r > R$

$\Phi(r=R,\phi) = V\cos(m\phi)$

θ is not restricted, so,

$$\Phi(r,\theta) = C\cos(m\theta)\,\frac{1}{r^m}$$

$$\Phi(r=R,\theta) = V\cos(m\theta) = C\cos(m\theta)\,\frac{1}{R^m} \Rightarrow C = VR^m$$

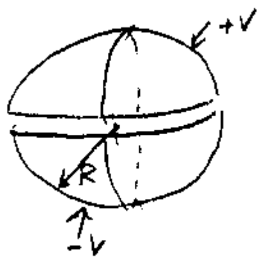
So,

$$\boxed{\Phi(r,\theta) = V\cos(m\theta)\left(\frac{R}{r}\right)^m} \qquad m \neq 0$$

for $m=0$
θ is restricted

$$\boxed{\Phi(r,\theta) = V\,\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)}}$$

10) $m=0$ problem



$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\cos\theta)$$

for $r < R$, $b_l = 0$

$$\Phi(r=R,\theta) = \begin{cases} +V & \text{for } 0 \le \theta \le \pi/2 \\ -V & \pi/2 \le \theta \le \pi \end{cases}$$

$$\Phi(r=R,x) = \sum_l C_l P_l(x) \quad , x = \cos\theta$$

$$C_l = \frac{2l+1}{2}\int_{-1}^{+1} dx\, \underbrace{\Phi(r=R,x)}_{\text{odd}}\, P_l(x)$$

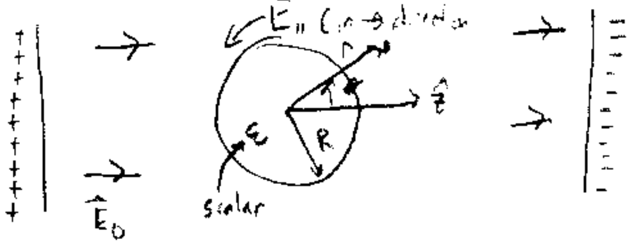
$\Rightarrow$ odd values of $P_l(x)$ contribute b/c limits are from -1 to $+1$

$$C_l = (2l+1)\int_0^1 dx\, V\, P_l(x) \qquad \text{for } l = 1, 3, 5, \ldots$$

$$\Phi(r,\theta) = V\sum_{l=0}^{\infty}\frac{C_l}{V}\left(\frac{r}{R}\right)^l P_l(\cos\theta)$$

note: for $m=1$ disturbance, can only have $l=0$ or 1

11) Dielectric in presence of uniform $\vec{E}$



angle α (azimuthal) coming out of page

azimuthal symmetry $\Rightarrow$ $\Phi = \sum_{l=0}^{\infty} \left(a_l r^l + \frac{b_l}{r^{l+1}} \right) P_l(\cos\theta)$

$r < R$ $\qquad \Phi_< = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta)$

$r > R$ $\qquad$ as $r \to \infty$, $\Phi_> \to -E_0 z$ $\quad$ (b/c $\nabla \Phi \to E_0 \hat{z}$) $\quad$, $z = r\cos\theta$

$\therefore\ a_l = 0\ \forall\, l \neq 1$ $\quad$, $\Phi_> = a_1 r \cos\theta + \sum_l \frac{b_l}{r^{l+1}} P_l(\cos\theta)$

B.C.

$\Delta E_{\parallel} = 0$ $\qquad \left.\frac{\partial \Phi_>}{\partial \theta}\right|_{r=R} = \left.\frac{\partial \Phi_<}{\partial \theta}\right|_{r=R}$

$\Delta D_{\perp} = 4\pi \sigma_{Free} = 0$ $\qquad$, but $(D_{\perp})_{vacuum} = (D_{\perp})_{\epsilon}$

$\underset{\epsilon = 1}{\uparrow}$

$$-\frac{\partial}{\partial r} \Phi_> \Big|_{r=R} = -\varepsilon \frac{\partial}{\partial r} \Phi_< \Big|_{r=R}$$

→ because of azimuthal symmetry, sphere does not mix (scatter) the l number!

So,

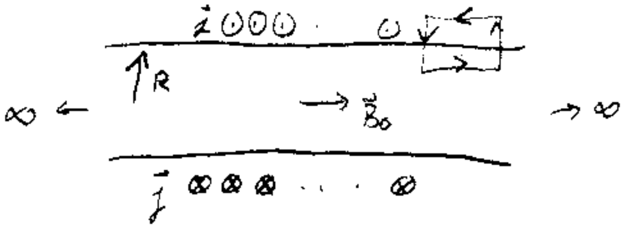
$$\Phi_> = -E_0 r \cos\theta + \frac{R^3(\varepsilon - 1)}{\varepsilon + 2} E_0 \left(\frac{1}{r^2} \right) \cos\theta$$

$$\Phi_< = \frac{-3E_0}{\varepsilon + 2} r \cos\theta$$

potential due to a dipole

$\Phi_p \Rightarrow \frac{R \cdot \vec{r}}{r^2}$ $\qquad \vec{P} = \frac{\varepsilon - 1}{\varepsilon + 2} R^2 E_0 \hat{z}$

12) calculate $\vec{A}$ for ∞ long Solenoid $\quad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}$, $\vec{B} = \nabla \times \vec{A}$



$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I \quad , \vec{j} = \lambda\,\delta(r-R)\,\hat{\theta}$$

$$\Rightarrow \left[\nabla \times (\nabla \times \vec{A})\right]_\theta = \frac{4\pi\lambda}{c}\,\delta(r-R)$$

where
$$\left[\nabla \times (\nabla \times \vec{A})\right]_\theta = -\frac{\partial^2}{\partial r^2}A_\theta - \frac{1}{r}\frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r^2}$$

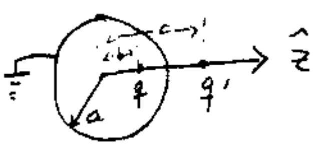
but laplacian in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}A_\theta\right) = \frac{\partial^2}{\partial r^2}A_\theta + \frac{1}{r}\frac{\partial}{\partial r}A_\theta$$

$$A_\theta(r) = \begin{cases} \frac{2\pi\lambda}{c}r & r<R \\ \frac{2\pi\lambda}{c}\frac{R^2}{r} & r>R \end{cases} \qquad B_z = \frac{2\pi\lambda}{c}\begin{cases} 2 & r<R \\ 0 & r>R \end{cases}$$

note: $\oint_C \vec{A}\,d\vec{l} = \int (\nabla\times\vec{A})\cdot d\vec{s} = \int \vec{B}\cdot d\vec{s} = \Phi_B$

13)



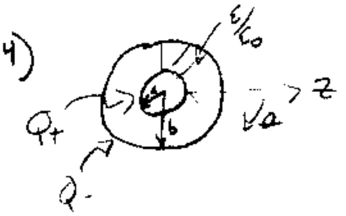
$$\Phi(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-b)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z-c)^2}}$$

note: $x^2+y^2+(z-b)^2 = x^2+y^2+z^2+b^2-2zb = r^2+b^2-2zb$, $z = r\cos\theta$

use B.C. $\Phi(r=a,\theta=0)=0$; $\Phi(r=a,\theta=\pi)=0$

$\therefore\ c = \frac{a^2}{b}$, $q' = -q\left(\frac{a}{b}\right)$; $\sigma = \frac{-1}{4\pi}\frac{\partial\Phi}{\partial r}\Big|_{r=a}$

14)

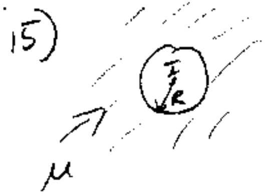


$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right]P_l(\cos\theta) \quad \text{or } \int dV\,\nabla\cdot D = 4\pi Q$$

no θ-dependence $\Rightarrow$ $l=0$ term is only surviving term

B.C. $\frac{\varepsilon}{\varepsilon_0}\frac{\partial\Phi}{\partial r}\Big|_{r=a} = \frac{\partial\Phi}{\partial r}\Big|_{r=a}$; $\vec{E} = -\nabla\Phi = \frac{Q}{4\pi r^2}\hat{r}\begin{cases}\frac{1}{\varepsilon/\varepsilon_0+1} & 0<\theta<\frac{\pi}{2} \\ \frac{1}{\varepsilon/\varepsilon_0+1} & \frac{\pi}{2}<\theta\end{cases}$

15)

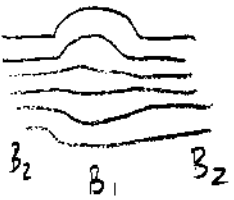


$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J}_{free} \Rightarrow \vec{H} = \frac{2I}{cr} \hat{\phi} \qquad 0 < r < \infty$$

$$\vec{B} = \vec{H} \quad r < R \quad (\mu = 1) \qquad , \quad \vec{B} = \mu \vec{H} \quad r > R$$

16) mirror machine

$$v_\perp = \sqrt{\frac{2 |\vec{m}| B_1}{M}} \qquad U_{total} = |\vec{m}| \, B_2$$



$$v_\parallel = \sqrt{\frac{2}{M} \left[U_{total} - |\vec{m}| B_1 \right]}$$

17) adiabatically invariant $\Rightarrow \oint p \, dq = \text{constant}$