E & M memorization sheet

Poisson's eq. $\nabla\cdot\vec{E} = 4\pi\rho$ no name: $\nabla\cdot\vec{B} = 0$

Faraday's law: $\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}$ Ampère's law: $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{E}$

} spatial and temporal scales are not set by Maxwell's eqs

moving charge: $\vec{E} = \dfrac{\vec{v}\times\vec{B}}{c}$, $\vec{v}_{drift} = c\,\dfrac{\vec{E}\times\vec{B}}{B^2}$

structure of Conservation Laws:

$$\frac{\partial}{\partial t}\Psi + \nabla\cdot\vec{J} = S$$

Ψ - scalar quantity of interest ... "stuff"
$\vec{J}$ - current of stuff (flow of Ψ)
S - local source or sink of stuff

conservation of charge: $\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} = 0$

conservation of Energy:

$$\frac{\partial}{\partial t}\Big[\underbrace{\frac{|\vec{E}|^2 + |\vec{B}|^2}{8\pi}}_{\text{energy}}\Big] + \nabla\cdot\Big[\underbrace{\frac{c}{4\pi}(\vec{E}\times\vec{B})}_{\text{poynting vector}}\Big] = \underbrace{-\vec{E}\cdot\vec{j}}_{\text{ohmic power}}$$

$$= -\frac{d}{dt}\left(\frac{1}{2}\rho_m v^2\right) \equiv u_m$$

$$\Rightarrow \frac{d}{dt}\left[u_E + u_B + u_m\right] + \nabla\cdot\vec{S} = 0$$

$$(\nabla\times\vec{B})\times\vec{B} = -\underbrace{\nabla\left(\frac{|\vec{B}|^2}{2}\right)}_{\substack{\text{pressure}\\ \text{(compression)}}} + \underbrace{(\vec{B}\cdot\nabla)\vec{B}}_{\substack{\text{tension}\\ \text{convective term}}}$$

conservation of momentum:

$$\underbrace{\frac{d}{dt}(\rho_m\vec{v})}_{\substack{\text{material}\\ \text{momentum density}}} + \frac{\partial}{\partial t}\underbrace{\left[\frac{\vec{S}}{c^2}\right]}_{\substack{\text{e\&m momentum}\\ \text{density}}} = \underbrace{\nabla\cdot\overleftrightarrow{T}}_{\substack{\text{Force}\\ \text{per unit}\\ \text{volume}}}$$

$$\Rightarrow \frac{d}{dt}\Big[\underset{\underset{\rho_m\vec{v}}{\parallel}}{\vec{p}_{mech}} + \underset{\underset{\frac{1}{4\pi c}(\vec{E}\times\vec{B})}{\parallel}}{\vec{p}_{em}}\Big] = \underset{\underset{\nabla\cdot\overleftrightarrow{T}}{\parallel}}{\vec{f}_{em}}$$

$$T_{\alpha\beta} = \frac{1}{4\pi}\left[E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2)\right]$$

Vector wave equation:

$$\frac{\partial^2\vec{E}}{\partial t^2} - c^2\nabla^2\vec{E} = 4\pi\frac{\partial}{\partial t}\vec{j} + 4\pi c^2(\nabla\rho)$$

$$\frac{\partial^2\vec{B}}{\partial t^2} - c^2\nabla^2\vec{B} = 4\pi c(\nabla\times\vec{j})$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\boxed{\vec{E} = -\nabla \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}}$$

$$\boxed{\frac{\partial^2 \vec{A}}{\partial t^2} - c^2 \nabla^2 \vec{A} = 4\pi c \vec{j} - c\nabla\left[\frac{\partial \phi}{\partial t} + c\nabla\cdot\vec{A}\right]}$$

$$\boxed{\nabla^2 \phi = -4\pi \rho - \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\vec{A}}$$

, coulomb gauge : $\nabla \cdot \vec{A} = 0$

Lorentz gauge: $\frac{\partial \phi}{\partial t} + c\nabla\cdot\vec{A} = 0$

canonical momentum eq:

$$\frac{d}{dt}\Big[\underbrace{\vec{p} + \frac{q}{c}\vec{A}}_{\equiv \vec{P}_{canonical}}\Big] = -q\nabla\Big[\underbrace{\phi - \frac{\vec{v}\cdot\vec{A}}{c}}_{\equiv \phi_{effective}}\Big] \Rightarrow H = \frac{(\vec{P}_{can} - \frac{q}{c}\vec{A})^2}{2m} + q\phi$$

3 basic properties of electric dipole moment: (1) energy, (2) Force, and (3) torque:

1) $U = -\vec{p}\cdot\vec{E}$
2) $\vec{F} = \vec{p}\cdot\nabla\vec{E}$
3) $\vec{\tau} = \vec{p}\times\vec{E}$

For magnetic systems

$$U = -\vec{m}\cdot\vec{B}$$
$$\vec{F} = \vec{m}\cdot\nabla\vec{B}$$
$$\vec{\tau} = \vec{m}\times\vec{B}$$

Boundary conditions: $\Delta E_{\parallel} = 0$ $\quad \Delta E_{\perp} = 4\pi\sigma_{total}$

$\Delta D_{\perp} = 4\pi\sigma_{free}$ $\quad \Delta P_{\perp} = -\sigma_{bound}$ or polarization

$\vec{H} = \vec{B} - 4\pi\vec{M}$, $\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J}_{conduction}$, $\vec{B} = \vec{\mu}\cdot\vec{H}$

Bessel's eqn: $\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f\right) + \left(k^2 - \frac{\nu^2}{r^2}\right)f = 0$

Soln. $J_\nu, Y_\nu \to$ standing waves

$H_\nu^{(1)}, H_\nu^{(2)} \to$ travelling waves

Green's Function in cylindrical coordinates

$$\left.\begin{matrix} G_>(\rho;\rho') \\ G_<(\rho;\rho') \end{matrix}\right\} = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^2} e^{im(\theta-\theta')} e^{ik(z-z')} (4\pi) \left\{\begin{matrix} K_m(k\rho)\, I_m(k\rho') \\ I_m(k\rho)\, K_m(k\rho') \end{matrix}\right.$$

note: spherical solution to Laplace's eqn: Legendre Poly has the orthogonal relation

$$\int_{-1}^{+1} dx\, P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1} \delta_{\ell';\ell}$$

if problem is symmetric only even $P_\ell(x)$ are non-zero, if antisymmetric only odd P_ℓ are non zero.

$$P_\ell(x) = \begin{cases} \text{even} & \ell \text{ even} \\ \text{odd} & \ell \text{ odd} \end{cases}$$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

→ single charges are diamagnetic

→ magnetization current density: $\vec{j}_m = c\,\nabla \times \vec{M}$ bound charge: $\rho_b = -\nabla \cdot \vec{P}$

→ surface current: $\vec{I} = c\,\vec{M} \times \vec{n}$ surface charge: $\sigma_b = \vec{P} \cdot \hat{n}$

u_{total} $\dfrac{\vec{H} \cdot \vec{B}}{8\pi}$, $\dfrac{\vec{D} \cdot \vec{E}}{8\pi}$, $\vec{D} = \vec{E} + 4\pi \vec{P}$

Force density $\dfrac{d\vec{F}}{dV} = \vec{M} \cdot \nabla \vec{B} = \left(\dfrac{\mu - 1}{4\pi}\right) \vec{H} \cdot \nabla \vec{B}$, $\vec{M} = \dfrac{(\mu-1)\vec{H}}{4\pi}$

→ $K = \pm \dfrac{\omega}{c}$ in vacuum, $K = \pm \dfrac{\omega}{c}\sqrt{\epsilon(\omega)}$ otherwise

$\vec{S} \equiv \dfrac{c}{4\pi} \vec{\tilde{E}} \times \vec{\tilde{B}}^*$ → circulating poynting flux (attached to object)

$\mathcal{P} \equiv \vec{\tilde{j}} \cdot \vec{\tilde{E}}^*$ → 'reversible' mechanical energy

$$\nabla \cdot (\mathrm{Im}\, \vec{S}) = -\mathrm{Im}\, \mathcal{P} + \frac{\omega}{4\pi} \left[|\vec{\tilde{B}}|^2 - |\vec{\tilde{E}}|^2 \right]$$

dielectric coefficient

$$\epsilon(\omega) = 1 + \frac{4\pi N \alpha(\omega)}{1 - \beta N \alpha(\omega)} \quad , \quad \alpha(\omega) = \sum_l \frac{F_l}{\omega_l^2 - \omega^2 - i\gamma_l \omega}$$

$$v_g(\omega) = \frac{v_p(\omega)}{1 + \omega \frac{\partial}{\partial \omega} \ln \sqrt{\epsilon_R}}$$

From Faraday's law can get relationship between $\vec{E}$ & $\vec{B}$ $\therefore$ $\vec{K} \times \tilde{\vec{E}} = \frac{\omega}{c} \tilde{\vec{B}}$

$$\frac{|\tilde{\vec{B}}|^2}{|\tilde{\vec{E}}|^2} = \epsilon_R \left[1 + \frac{1}{4} \left(\frac{\epsilon_I}{\epsilon_R} \right)^2 \right]$$

AC conductance $\rightarrow$ $\omega \epsilon_I$
AC reactance $\rightarrow$ $\omega(1 - \epsilon_R)$

For conducting media: $\epsilon = 1 + i \frac{4\pi\sigma}{\omega}$

time averaged power: $\langle dP \rangle = r^2 d\Omega \langle \vec{S} \cdot \hat{r} \rangle$

per solid angle $\langle \frac{dP}{d\Omega} \rangle = r^2 \langle \vec{S} \cdot \hat{r} \rangle$

radiation resistance of antenna: $R_{rad} = \dfrac{\int d\Omega \langle \frac{dP}{d\Omega} \rangle}{\frac{1}{2} I^2}$

Lienard-Wiechert Potentials:

$$\vec{A}(\vec{r}, t) = q \left[\frac{\vec{\beta}}{R - \vec{R} \cdot \vec{\beta}} \right]_{\text{retarded time}}$$

$$\phi(\vec{r}, t) = q \left[\frac{1}{R - \vec{R} \cdot \vec{\beta}} \right]_{\text{retarded time}}$$

$$\vec{S}_a = \frac{q^2}{4\pi c^3 R^2} |\vec{a}|^2 \sin^2\theta \, \hat{R}$$

, Larmor radiation Formula : $P = \frac{2}{3} \frac{q^2}{c^3} |\vec{a}|^2$

$$\vec{a} = \frac{\ddot{\vec{p}}}{q}$$

$$\langle P \rangle = \frac{\omega^4}{3c^3} |\tilde{\vec{p}}|^2$$

B.C.

$$E^{\perp}_{above} - E^{\perp}_{below} = 4\pi\sigma$$

$$E^{\parallel}_{above} = E^{\parallel}_{below} = 0$$

$$B^{\perp}_{above} = B^{\perp}_{below}$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \frac{4\pi}{c} K \ , \ K = \frac{I_{enc}}{l}$$

* separation of variables. If a direction is unbounded like z, you can use a Fourier transform in z to represent the potential. For ex (if you have θ symmetry)

$$\Phi(\rho, z) = \int_{-\infty}^{\infty} e^{ikz} \frac{dk}{2\pi} \tilde{\Phi}(\rho, k)$$

- $\vec{F} = \nabla(\vec{m} \cdot \vec{B})$, $|\vec{m}| = \frac{I}{c} \times (\text{area})$

- Drude model

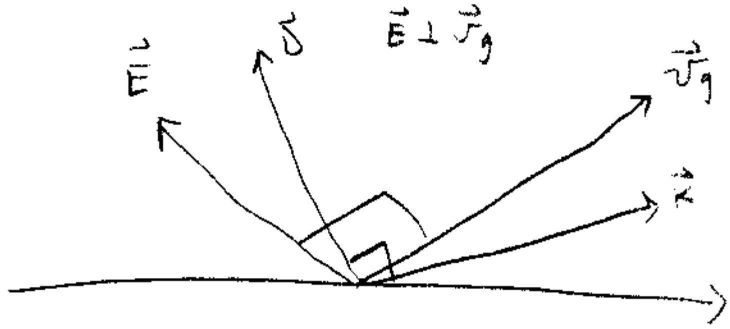
$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} + i\frac{\omega_p^2 \omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

- radiation

$$\vec{A} = \frac{e^{ikr}}{cr} \int \vec{J}(\vec{r}') e^{-i\vec{k}\cdot\vec{r}'} d^3r'$$

$\vec{D} \perp \vec{k}$

$\vec{E} \perp \vec{v}_g$



- retarded time t'

$$t' = t - \frac{r}{c}$$