Spring 2000 #10

Consider the Hamiltonian of a 1-D harmonic oscillator to be $-(d/dx)^2 + x^2$. Take it to be known that the spectrum is discrete.

a) prove that the eigenvalues are positive

(Refer to Shankar pg 189)

Rewrite Hamiltonian as $H = \hat{p}^2 + \hat{x}^2$ where $\hat{p}, \hat{x}$ are operators such that

$$\langle x|\hat{p}|\psi\rangle = -i\frac{d}{dx}\psi(x) \quad ; \quad \langle x|\hat{x}|\psi\rangle = x\psi(x)$$

Next evaluate $\langle H\rangle = \langle\psi|H|\psi\rangle = E$

$$\langle H\rangle = \langle\psi|\hat{p}^2|\psi\rangle + \langle\psi|\hat{x}^2|\psi\rangle$$

$$= \langle\psi|\hat{p}^+\hat{p}|\psi\rangle + \langle\psi|\hat{x}^+\hat{x}|\psi\rangle$$

$$= \langle\hat{p}\psi|\hat{p}\psi\rangle + \langle\hat{x}\psi|\hat{x}\psi\rangle$$

since the norm of any nonzero state is positive, $\langle H\rangle$ and hence E are positive for all eigenstates.

b) Determine the wave function and the energy of the ground state. (Refer to Abers pgs 58-59)

→ Create two new operators:

$$a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \qquad a^+ = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$$

One can show that $[a, a^+] = 1$

the Hamiltonian can be rewritten by noting that

$$\hat{x} = \frac{1}{\sqrt{2}}(a + a^+) \qquad \hat{p} = \frac{-i}{\sqrt{2}}(a - a^+)$$

$$H = -\frac{1}{2}(a - a^+)^2 + \frac{1}{2}(a + a^+)^2$$

simplification leads to

$$H = \frac{1}{2}(aa^+ + a^+a)$$

use relation $[a, a^+] = 1 = aa^+ - a^+a$

$$H = \frac{1}{2}(1 + a^+a + a^+a)$$

$$H = \left(a^+a + \frac{1}{2}\right)$$

Next find $[a, a^\dagger a]$

$$[a, a^\dagger a] = a^\dagger [a, a] + [a, a^\dagger] a = a$$

so

$$[a, H] = aH - Ha = a \quad \Rightarrow Ha = aH - a$$

let Ha act on eigenstate of H

$$Ha|\psi\rangle = aH|\psi\rangle - a|\psi\rangle = (E_n - 1)a|\psi\rangle$$

Thus a acts a lowering operator

To find lowest state, use fact that $a|\psi_0\rangle = 0$

Thus

$$\langle\psi_0|a^\dagger a|\psi_0\rangle = 0$$

Substitute $a^\dagger a = H - \frac{1}{2}$

$$\langle\psi_0|(H - \tfrac{1}{2})|\psi_0\rangle = 0$$

$$E_0 - \frac{1}{2} = 0$$

Thus, the ground state energy is $\boxed{E_0 = \frac{1}{2}}$

to find wave function, use $a|\psi_0\rangle = 0$ again

$$\cancel{\frac{1}{\sqrt{2}}}\langle x|(\hat{x} + i\hat{p})|\psi_0\rangle = 0 \qquad \text{(work in } x \text{ basis)}$$

$$x\psi_0(x) + \frac{d}{dx}\psi_0(x) = 0$$

$$\frac{d}{dx}\psi_0(x) = -x\psi_0(x)$$

$$\frac{d\psi_0(x)}{\psi_0(x)} = -x\,dx \qquad \Rightarrow \ln\psi_0(x) = \frac{-x^2}{2} + C$$

$$\psi_0(x) = Ae^{\frac{-x^2}{2}} \qquad A^{-2} = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\boxed{\psi_0(x) = \pi^{-\frac{1}{4}} e^{\frac{-x^2}{2}}}$$