Spring 2000

10) Given the Hamiltonian of a harmonic oscillator

$$H = -\left(\frac{d}{dx}\right)^2 + x^2$$

with the understanding that the spectrum is discrete,

i) prove the eigenvalues are positive

ii) determine the ground state wave function & energy.

Solution: We recognise that this Hamiltonian is double the Hamiltonian of the standard S.H.O.

$$H \Rightarrow 2\, H_{SHO} = \frac{p_x^2}{M} + M\omega^2 x^2$$

with ω, M & $\hbar = 1$

We can therefore use the standard representation for H_{SHO} of the raising & lowering operators

$$a = \frac{1}{\sqrt{2}}(x + ip) \quad ; \quad a^{+} = \frac{1}{\sqrt{2}}(x - ip)$$

and $H_{SHO} = a^{+}a + \frac{1}{2}$.

for $\psi_0(x)$ being the ground state

$$a\,\psi_0(x) = 0$$

so $H_{SHO}\psi_0(x) = (E_0)^{SHO}\psi_0(x)$

$$\left(a^{+}a + \frac{1}{2}\right)\psi_0(x) = (E_0)^{SHO}\psi_0(x)$$

so $(E_0)^{SHO} = \frac{1}{2}$

from $\langle H\rangle = \langle 2H_{SHO}\rangle = 2E_0^{SHO}$

$$\boxed{E_0 = 1}$$

using the canonical representation of the 1-D momentum in X-space

$$p_x \to -i \frac{d}{dx}$$

$$0 = a\,\psi_0(x) = \frac{1}{\sqrt{2}}\,(x + i p_x)\,\psi_0(x)$$

$$\text{so} \quad x\,\psi_0(x) = -\frac{d}{dx}\psi_0(x)$$

$$\Rightarrow \psi_0(x) = A\,e^{-\frac{x^2}{2}}$$

$$\text{Normalizing:} \quad 1 = \int_{-\infty}^{\infty} |A|^2 |\psi_0(x)|^2\,dx$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= |A|^2 \sqrt{\pi}$$

so

$$\boxed{\psi_0(x) = \frac{1}{\pi^{1/4}}\,e^{-\frac{x^2}{2}}}$$

i) $\langle H \rangle = \langle p^2 \rangle + \langle x^2 \rangle$

$$= \langle \psi | p^\dagger p | \psi \rangle + \langle \psi | x^\dagger x | \psi \rangle$$

$$= \langle \psi'_p | \psi'_p \rangle + \langle \psi''_x | \psi''_x \rangle$$

where $p|\psi\rangle = |\psi'_p\rangle$, $x|\psi\rangle = |\psi''_x\rangle$

Standard-Normalization of these wavefn's is positive so $H \geq 0$