Spring 2000 #11

A simple model for the dynamics of an atom near an impenetrable wall is obtained by considering a particle in the presence of the following one-dimensional potential:

$$V(x) = \begin{cases} +\infty & x \le 0 \\ -V_0\,\delta(x-d) & x \ge 0 \end{cases}$$

with $V_0 > 0$ and $d > 0$.

(a) obtain a simple transcendental equation for the energy of a bound state.

→ for $0<x<d$, $x>d$, the Schrödinger equation is:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x) \quad \Rightarrow \quad \frac{d^2\psi}{dx^2} = k^2\psi(x)$$

$$k = \sqrt{\frac{-2mE}{\hbar^2}}$$

thus $\psi = \begin{cases} Ae^{kx} - Ae^{-kx} = 2A\sinh(kx) & 0<x<d \\ Ce^{-kx} & x>d \end{cases}$

continuity of ψ at $x=d$ yields

$$2A\sinh(kd) = Ce^{-kd} \qquad (i)$$

integration over discontinuity yields:

$$-\frac{\hbar^2}{2m}\left[\left.\frac{d\psi}{dx}\right|_{x=d^+} - \left.\frac{d\psi}{dx}\right|_{x=d^-}\right] - V_0\psi(d) = 0$$

$$-\frac{\hbar^2}{2m}\left(-kCe^{-kd} - k\,2A\cosh(kd)\right) = V_0 Ce^{-kd} \qquad (ii)$$

substitute (i) into (ii) to get

$$\frac{\hbar^2 k}{2m}\left(\cancel{2A}\sinh(kd) + \cancel{2A}\cosh(kd)\right) = V_0\,\cancel{2A}\sinh(kd)$$

$$\boxed{\left[1+\coth(kd)\right] = \frac{2mV_0}{\hbar^2 k}}$$

b) Solve this equation to leading non-trivial dependence in d, when the wall is far away from the atom.

let $d \gg \frac{1}{k}$, so $\coth(kd) = \frac{e^{kd}+e^{-kd}}{e^{kd}-e^{-kd}} \approx 1$

thus $\quad (l+1) = \dfrac{2mV_0}{\hbar^2 k}$

or $\quad k = \sqrt{-\dfrac{2mE}{\hbar^2}} = \dfrac{mV_0}{\hbar^2}$

$$-\frac{2mE}{\hbar^2} = \frac{m^2 V_0^2}{\hbar^4} \quad \Rightarrow \quad \boxed{E = -\frac{V_0^2 m}{2\hbar^2}}$$

c) What is the exact condition on V_0 and d for the existence of at least one bound state?

let $y = kd$, then the transcendential equation becomes

$$\coth(y) = \frac{2mV_0 d}{\hbar^2 y} - 1$$

now since $y > 1$, $\coth(y) > 1$, thus to find minimal V_0, need to look at $\coth(y)$ for small y.

$$\coth(y) \approx \frac{1}{y} = \frac{2mV_0 d}{\hbar^2 y} - 1$$

$$1 = \frac{2mV_0 d}{\hbar^2} - y \quad \Rightarrow \quad y = \frac{2mV_0 d}{\hbar^2} - 1$$

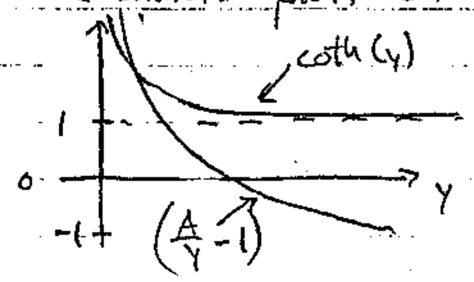
we know $y = kd > 0$, so $\dfrac{2mV_0 d}{\hbar^2} > 1$

or

$$\boxed{V_0 > \frac{\hbar^2}{2md}}$$

d) Can the system ever have more than one bound state?

→ examine plots of $\coth(y)$, $\dfrac{2mV_0 d}{\hbar^2 y} - 1$



the plots can only cross once, so $\boxed{\text{there can be only one bound state}}$