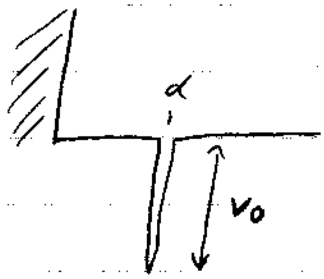
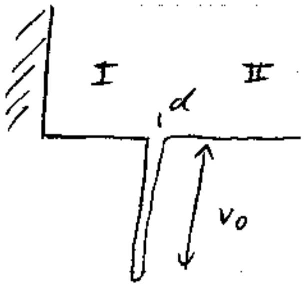
# QM S'00 #11

$$V(x) = \begin{cases} +\infty & x \le 0 \\ -V_0\,\delta(x-a) & x \ge 0 \end{cases}$$




$V_0 > 0\,;\; a > 0$

a) $E < 0$




I) $V = 0 \;\to\; -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \;\Rightarrow\; \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi \;;\; k = \frac{\sqrt{-2mE}}{\hbar} \ge 0$ as $E < 0$

$\Rightarrow \frac{d^2\psi}{dx^2} = k^2\psi \Rightarrow \psi(x) = A\sinh(kx) + B\cosh(kx)$

Now $\psi(x=0) = 0 \Rightarrow B = 0 \quad \therefore \quad \psi(x) = A\sinh(kx)$

II) $V = 0 \;\to\; -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi \;\Rightarrow\; \frac{d^2}{dx^2}\psi = -\frac{2mE}{\hbar^2}\psi \;;\; k = \frac{\sqrt{-2mE}}{\hbar}$

$\to \frac{d^2\psi}{dx^2} = k^2\psi \Rightarrow \psi(x) = Ce^{kx} + De^{-kx}$

Now $\psi(x \to \infty) = 0 \Rightarrow C = 0 \quad \therefore \quad \psi(x) = De^{-kx}$

So in summary

$$\psi(x) = \begin{cases} A\sinh(kx) & 0 < x < a \\ De^{-kx} & x > a \end{cases}$$

Now to apply the B.C.

$x = d$

$$A\sinh(kd) = De^{-kd} \quad (1)$$

and the derivative at $d$:

$$\int_{d-\epsilon}^{d+\epsilon} dx \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi - V_0\int_{d-\epsilon}^{d+\epsilon}\psi\,\delta(x-d)\,dx = \cancel{\int_{d-\epsilon}^{d+\epsilon} E\psi\,dx} \to 0$$

$$\Rightarrow \frac{-\hbar^2}{2m}\frac{d}{dx}\psi\Big|_{d-\epsilon}^{d+\epsilon} = V_0 De^{-kd} \Rightarrow -kDe^{-kd} - Ak\cosh(kd) = -\frac{2mV_0}{\hbar^2}De^{-kd} \quad (2)$$

$(1) \to (2)$ $$k\cancel{A}\sinh(kd) + k\cancel{A}\cosh(kd) = \frac{2mV_0}{\hbar^2}\cancel{A}\sinh(kd)$$

$$\sinh(kd) + \cosh(kd) = \frac{2mV_0}{\hbar^2 k}\sinh(kd)$$

$$\frac{e^{kd} \cancel{- e^{-kd}} + e^{kd} \cancel{+ e^{-kd}}}{2} = \frac{2mV_0}{\hbar^2 k}\sinh(kd)$$

$$\frac{\cancel{2}e^{kd}}{\cancel{2}} = \frac{\cancel{2}mV_0}{\hbar^2 k}\sinh(kd) = \frac{mV_0}{\hbar^2 k}\frac{(e^{kd} - e^{-kd})}{\cancel{2}}$$

$$\Rightarrow \boxed{k = \frac{mV_0}{\hbar^2}\left(1 - e^{-2kd}\right)}$$

b) For $d \to \infty$ $k_\infty = \frac{mV_0}{\hbar^2}$, so change the above to an iterative formula:

$$k_{n+1} = k_\infty\left(1 - e^{-2k_n d}\right)$$

try $k_0 = k_\infty \Rightarrow k_1 = k_\infty\left(1 - e^{-2k_\infty d}\right)$

$$\frac{1}{e^2} \approx \frac{1}{(2.72)^2} \approx \frac{1}{\left(\frac{11}{4}\right)^2} = \frac{16}{121} \approx \frac{20}{120} = \frac{2}{12} = 1/6 \approx 0.16$$

$2.75 = \frac{11}{4}$

now for $d = \frac{1}{k_\infty} \to k_1 = k_\infty\left(1 - e^{-2}\right)$

$$\approx k_\infty(0.84)$$

so $d > \frac{1}{k_\infty} = \frac{\hbar^2}{mV_0}$

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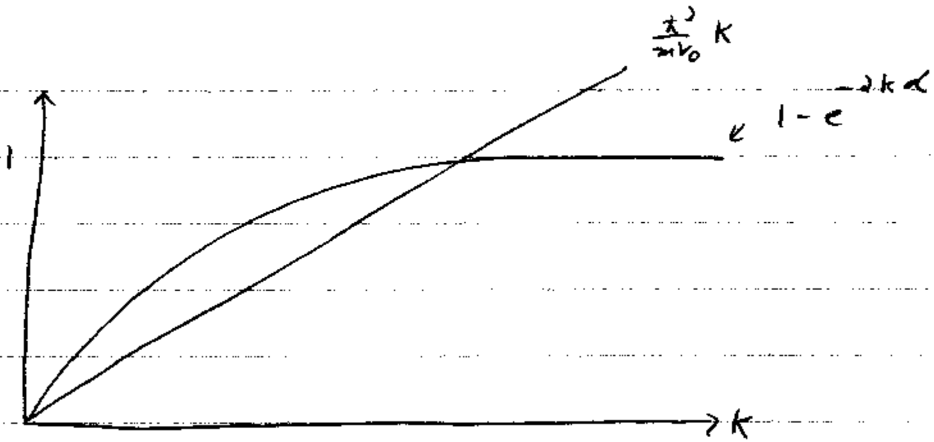
c)




The slope of $\frac{\hbar^2}{mV_0}k$ has to be less than $1-e^{-2k\alpha}$

$$\left.\frac{d}{dk}\right|_{k=0}\frac{\hbar^2}{mV_0}k < \left.\frac{d}{dk}\right|_{k=0}\left(1-e^{-2k\alpha}\right) \Rightarrow \frac{\hbar^2}{mV_0} < (-1)(-2\alpha)\underbrace{e^{-2\alpha\cdot 0}}_{=1}$$

so $\boxed{\alpha > \frac{\hbar^2}{2mV_0}}$

d) No, there is only one possible intersection.