Consider the Hamiltonian for a general top with principal moments of inertia I_1, I_2, I_3.

(a) For the symmetric top with $I = I_1 = I_2 \neq I_3$, derive all the energy levels

The general Hamiltonian for this system is given by (see Spring 2000 # 6)

$$H = \frac{\hbar^2}{2I_1} J_x^2 + \frac{\hbar^2}{2I_2} J_y^2 + \frac{\hbar^2}{2I_3} J_z^2$$

set $\hbar = 1$ and use what we are given that $I = I_1 = I_2 \neq I_3$. That is,

$$H = \frac{J_x^2 + J_y^2}{2I} + \frac{J_z^2}{2I_3} \qquad (1)$$

We know that the J_i's must satisfy the following commutation relation:

$$[J_i, J_j] = i \sum_k \varepsilon_{ijk} J_k$$

From here we can construct the algebra in the basis of eigenstates $|j, m\rangle$ of J^2 and J_z. See Abers pp 77-80. So, we construct raising and lowering operators J_+ & J_- such that

$$J_x = \frac{J_+ + J_-}{2} \quad \text{and} \quad J_y = \frac{J_+ - J_-}{2i} \qquad (2)$$

if you have trouble remembering these think of the definitions of $\cos x$ and $\sin x$

$$\cos x = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sin x = \frac{e^x - e^{-x}}{2i}$$

and note the similar form.

From eq (2), we know

$$J_x^2 + J_y^2 = \frac{1}{4}\Big(J_+^2 + J_-^2 + J_+J_- + J_-J_+ \underset{\text{from } i^2}{-} J_+^2 - J_-^2 + J_+J_- + J_-J_+ \Big)$$

$$= \frac{1}{4}\left(2J_+J_- + 2J_-J_+ \right)$$

$$\Rightarrow J_x^2 + J_y^2 = \frac{1}{2}\left(J_+J_- + J_-J_+ \right)$$

substitute these results into eq (1)

$$\boxed{H = \frac{1}{4I}\left(J_+J_- + J_-J_+ \right) + \frac{J_z^2}{2I_3}}$$

recall (Abers eq 3.113a & 3.113b)

$$J_z |j,m\rangle = m\,|j,m\rangle$$
$$J_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm1)}\;|j,m\pm1\rangle$$

so,

$$H|j,m\rangle = \left[\frac{1}{4I}(J_+J_- + J_-J_+) + \frac{J_z^2}{2I_3}\right]|j,m\rangle$$

$$= \frac{1}{4I} J_+ \sqrt{j(j+1)-m(m-1)}\;|j,m-1\rangle + J_-\sqrt{j(j+1)-m(m+1)}\;|j,m+1\rangle + \frac{m^2}{2I_3}|j,m\rangle$$

$$= \frac{1}{4I}\sqrt{j(j+1)-(m-1)m}\sqrt{j(j+1)-m(m-1)}\;|j,m\rangle + \frac{m^2}{2I_3}|j,m\rangle$$

$$+ \sqrt{j(j+1)-(m+1)m}\sqrt{j(j+1)-m(m+1)}\;|j,m\rangle$$

where $\sqrt{[j(j+1)-m(m\pm1)][j(j+1)-m(m\pm1)]} = \sqrt{[j(j+1)-m(m\pm1)]^2} = j(j+1)-m(m\pm1)$

so,

$$H|j,m\rangle = \frac{1}{4I}\left[j(j+1)-m(m-1) + j(j+1)-m(m+1)\right]|j,m\rangle + \frac{m^2}{2I_3}|j,m\rangle$$

$$= \frac{1}{2I}\left[j(j+1)-m^2\right]|j,m\rangle + \frac{m^2}{2I_3}|j,m\rangle$$

Thus,

$$\boxed{E_{jm} = \left[\frac{j(j+1)-m^2}{2I} + \frac{m^2}{2I_3}\right]}$$

(b) A slightly asymmetric top has $2\Delta = I_1 - I_2 \neq 0$, $2I = I_1 + I_2$ and $\Delta \ll I$, with $I \neq I_3$. Compute the $j=0$ and $j=1$ energies up to and including first order in Δ.

let's use what we know from part (a) and what we are given to make this as simple as possible.

Solve for I_2 from what was given:

$$I_2 = I_1 - 2\Delta$$

so the second term in the general Hamiltonian becomes

$$\frac{J_y^2}{2I_2} = \frac{J_y^2}{2(I_1-2\Delta)} = \frac{J_y^2}{2I_1\left(1-\frac{2\Delta}{I_1}\right)} \underset{\uparrow}{\simeq} \frac{J_y^2}{2I_1}\left(1+\frac{2\Delta}{I_1}\right)$$

binomial theorem
since $\Delta \ll I$

So, our Hamiltonian is

$$H = \frac{J_x^2}{2I_1} + \frac{J_y^2}{2I_1} + \frac{2\Delta}{2I_1^2}J_y^2 + \frac{J_z^2}{2I_3} = \frac{(J_x^2+J_y^2)}{2I_1} + \frac{J_z^2}{2I_3} + \frac{\Delta J_y^2}{I_1^2}$$

$$\Rightarrow H = H_a + \frac{\Delta J_y^2}{I_1^2}$$, where H_a = Hamiltonian from part (a) with $I \to I_1$

So, let's consider the "perturbed term" in the Hamiltonian

$$H'|j,m\rangle = \frac{\Delta}{I_1^2}J_y^2|j,m\rangle = \frac{\Delta}{I_1^2}\left(\frac{J_+-J_-}{2i}\right)^2|j,m\rangle = \frac{-\Delta}{4I_1^2}\left(J_+^2 + J_-^2 - J_+J_- - J_-J_+\right)|j,m\rangle$$

$$= \frac{-\Delta}{4I_1^2}\Big[J_+\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle + J_-\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle$$

$$+\left(j(j+1)-m(m-1) + j(j+1)-m(m+1)\right)|j,m\rangle\Big]$$

$$\Rightarrow \langle j,m|H'|j,m\rangle = +\frac{\Delta}{2I_1^2}\left[j(j+1)-m^2\right]$$

So, the energy is

$$\boxed{E_{jm} = \frac{j(j+1)-m^2}{2I_1} + \frac{\Delta j(j+1)}{2I_1^2} - \frac{\Delta m^2}{2I_1^2} + \frac{m^2}{2I_3}}$$

→ for $j=0$, $m=0$

$$\boxed{E_{00} = 0}$$

→ for $j=1$

- $m=+1$,

$$E_{11} = \frac{2-1}{2I_1} + \frac{2\Delta}{2I_1^2} - \frac{\Delta}{2I_1^2} + \frac{1}{2I_3} = \frac{1}{I_1} + \frac{1}{2I_3} + \frac{\Delta}{2I_1^2}$$

$$\therefore \boxed{E_{11} = \left(\frac{1}{I_1} + \frac{1}{2I_3}\right) + \underbrace{\frac{\Delta}{2I_1^2}}_{\text{perturbed part}}}$$

- $m=0$,

$$E_{10} = \frac{2}{2I_1} - \frac{\Delta 2}{2I_1^2} + 0 + 0 = \boxed{\frac{1}{I_1} - \frac{\Delta}{I_1^2}}$$

- $m=-1$

$$E_{1,-1} = \frac{2-1}{2I_1} + \frac{2\Delta}{2I_1^2} - \frac{\Delta}{2I_1^2} + \frac{1}{2I_3} = \boxed{E_{11}}$$