

Spring 2000 Solution 13

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1 Caveat

There is a typo in Part (b). Section (2) should read

$$(2) \text{ from } (p_2, V_2) \text{ to } (p_2, V_1)$$

or the process has a discontinuity.

2 Solution

2.1 Part (a)

Start with the First Law:

$$dE = dQ - dW = TdS - pdV \quad dS = \frac{1}{T}dE + \frac{p}{T}dV \quad (1)$$

Remember dW is defined to be the work done *by* the system, while dQ is the change in the system's heat.

The system is going from a known volume and temperature to a different specified volume and temperature. The work term is already expressed in terms of dV so it's fine. We have to expand the differential dE if we plan to integrate it. Since we know the Ideal Gas Law

$$PV = nkT = NRT = RT \quad (2)$$

where n is the number of particles, N is the number of moles (which is fixed at 1), a complete set of variables is 2: T and V . We expand dE in these:

$$dE = \left(\frac{dE}{dV}\right)_T dV + \left(\frac{dE}{dT}\right)_V dT \quad (3)$$

For an ideal gas the first term is zero. So the differential is just

$$dE = \left(\frac{dE}{dT}\right)_V dT = C_v dT \quad (4)$$

and, replacing p by $p = \frac{RT}{V}$:

$$\int_{(1)}^{(2)} dS = C_v \int_{T_1}^{T_2} \frac{dT}{T} + R \int_{V_1}^{V_2} \frac{dV}{V} \quad (5)$$

$$\Delta S = C_v \ln \frac{T_2}{T_1} + R \ln \frac{V_2}{V_1} \quad (6)$$

2.2 Part (b)

The efficiency of a process (closed; $\Delta E = 0$) is defined to be

$$\eta = \frac{\text{work done}}{\text{heat in}} = \frac{Q_{in} - Q_{out}}{Q_{in}} \quad (7)$$

where Q_{in} is the heat entering the system, and Q_{out} is the heat dumped into the low-temperature reservoir.

2.2.1 (1) Adiabatic expansion

Since "adiabatic" means no transfer of heat, Q_{in} and Q_{out} are zero.

2.2.2 (2) Compression at constant pressure

We go from (P_2, V_2) to (P_2, V_1) . Write down the First Law to get dQ :

$$dQ = dE + dW \quad \int dQ = \int dE + \int dW \quad (8)$$

From the Ideal Gas Law

$$dT = \frac{P_2}{R} dV \quad (9)$$

so

$$\int dE = \int C_v dT = C_v \int_{V_2}^{V_1} \frac{P_2}{R} dV = C_v \frac{P_2}{R} (V_1 - V_2) \quad (10)$$

The work term is just

$$\int dW = \int_{V_2}^{V_1} P_2 dV = P_2 (V_1 - V_2) \quad (11)$$

So the total heat change of the system is

$$\Delta Q = C_v \frac{P_2}{R} (V_1 - V_2) + P_2 (V_1 - V_2) = \left(C_v \frac{P_2}{R} + P_2 \right) (V_1 - V_2) \quad (12)$$

Using relation 5.7.17 from Reif

$$C_p - C_v = R \quad (13)$$

for an ideal gas,

$$\Delta Q = \frac{C_p P_2}{R} (V_1 - V_2) \quad (14)$$

So we identify $-\Delta Q$ as the heat dumped into the reservoir:

$$Q_{out} = \frac{C_p P_2}{R} (V_2 - V_1) \quad (15)$$

2.2.3 (3) Constant-Volume heat absorbtion

The volume does not change, so $\int dW = 0$. We write down the First Law:

$$\int dQ = \int dE = C_v \int dT \quad (16)$$

We are given pressure information, so use the Ideal Gas Law to express dT :

$$dT = \frac{V_1}{R} dP \quad (17)$$

therefore

$$\Delta Q = C_v \frac{V_1}{R} \int_{P_2}^{P_1} dP = C_v \frac{V_1}{R} (P_1 - P_2) = Q_{in} \quad (18)$$

2.2.4 Efficiency

With this information, we can write

$$\eta = \frac{Q_{in} - Q_{out}}{Q_{in}} = \frac{C_v \frac{V_1}{R} (P_1 - P_2) - \frac{C_p P_2}{R} (V_2 - V_1)}{C_v \frac{V_1}{R} (P_1 - P_2)} \quad (19)$$

Cancelling we get

$$\eta = 1 - \frac{C_p P_2 (V_2 - V_1)}{C_v V_1 (P_1 - P_2)} = 1 - \frac{C_p P_2 V_1 (\frac{V_2}{V_1} - 1)}{C_v V_1 P_2 (\frac{P_1}{P_2} - 1)} \quad (20)$$

$$\eta = 1 - \gamma \left(\frac{\frac{V_2}{V_1} - 1}{\frac{P_1}{P_2} - 1} \right) \quad \gamma = \frac{C_p}{C_v} \quad (21)$$