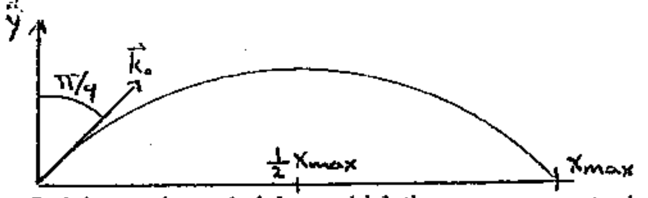
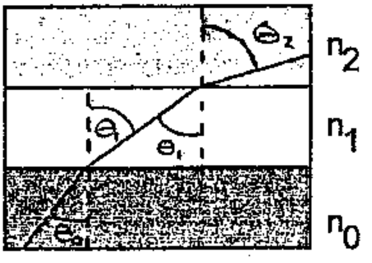
Spring 2000 #2

Consider the refraction of a plane electromagnetic wave in a nonuniform medium with an index of refraction $n(y) = 1 - y/L$. At y=0 the wave propagates at an angle $\theta = 45°$ with respect to the normal.



1) Using Snell's Law, find the maximum height at which the wave propagates horizontally (y_{max}).

First consider the case of three media:



Use of Snell's Law twice yields:

$$n_0 \sin\theta_0 = n_1 \sin\theta_1 = n_2 \sin\theta_2$$

The transitive property also applies in the continuous case, allowing one to write

$$n(0)\sin\left(\frac{\pi}{4}\right) = n(y_{max})\sin\left(\frac{\pi}{2}\right)$$

$$\frac{1}{\sqrt{2}} = \left(1 - \frac{y_{max}}{L}\right)$$

Solving for y_{max} yields:

$$y_{max} = L\left(1 - \frac{1}{\sqrt{2}}\right)$$

2) Find the horizontal range x_{max} where the wave has returned to y=0.

Start by finding $\frac{d\theta}{dx} = \left(\frac{d\theta}{dy}\right)\left(\frac{dy}{dx}\right)$

We already have the relation between θ and y:

$$\frac{1}{\sqrt{2}} = \left(1 - y/L\right)\sin\theta$$

$$y = L - \frac{L}{\sqrt{2}\sin\theta}$$

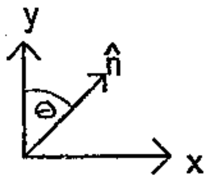
Differentiating yields:

$$dy = \frac{L\cos\theta}{\sqrt{2}\sin^2\theta}d\theta$$

or,

$$\frac{d\theta}{dy} = \frac{\sqrt{2}\sin^2\theta}{L\cos\theta}$$

To find dy/dx, consider a unit vector pointing in the direction of wave propagation:



Now $y = \cos\theta$ $x = \sin\theta$ so,

$$\frac{dy}{dx} = \frac{\cos\theta}{\sin\theta}$$

Thus,

$$\frac{d\theta}{dx} = \frac{\sqrt{2}\sin^2\theta}{L\cos\theta}\frac{\cos\theta}{\sin\theta} = \frac{\sqrt{2}}{L}\sin\theta$$

Rearrange and integrate to yield:

$$\int_{\pi/4}^{\pi/2}\frac{d\theta}{\sin\theta} = \frac{\sqrt{2}}{L}\int_0^{x\max/2}dx$$

$$\frac{1}{2}\ln\left[\frac{1-\cos\theta}{1+\cos\theta}\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2L}x_{\max}$$

$$\ln(1) - \ln\left(\frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}\right) = \frac{\sqrt{2}}{L}x_{\max}$$

$$x_{\max} = \frac{L}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$