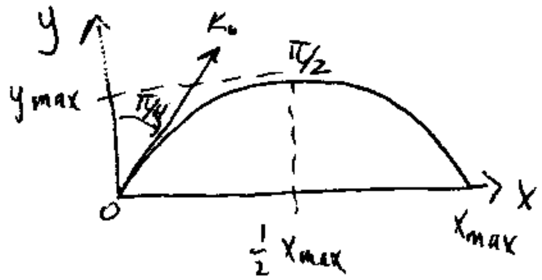
Spring 2000 #2 (p 1 of 2)

Consider the refraction of a plane electromagnetic wave in a nonuniform medium where the refractive index varies with height y as $n(y) = 1 - y/L$ with $L = \text{const}$. At $y = 0$ the wave propagates at an angle $\theta_0 = 45°$ wrt the normal.

(1) Using Snell's law find the maximum height at which the wave propagates horizontally, $y_{max} = ?$

(2) Find the horizontal range x_{max} where the wave has refracted back to $y = 0$.



(1)

Snell's law:

$$\underbrace{n(y=0)}_{=1}\ \underbrace{\sin(\theta = \tfrac{\pi}{4})}_{\frac{\sqrt{2}}{2}} = n(y = y_{max})\ \underbrace{\sin(\theta = \tfrac{\pi}{2})}_{=1}$$

$$\Rightarrow n(y = y_{max}) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow 1 - \frac{y_{max}}{L} = \frac{\sqrt{2}}{2} \qquad \therefore \boxed{y_{max} = L\left(1 - \frac{1}{\sqrt{2}}\right)}$$

(2)

Using the hint, we can start with an expression for $\frac{d\theta}{dx}$ and use this to find an expression for x_{max}.

From the chain rule, we have

$$\frac{d\theta}{dx} = \frac{d\theta}{dy}\frac{dy}{dx}, \quad \text{where } \frac{dx}{dy} = \tan\theta \quad \text{\& from Snell's law} \quad \frac{1}{\sqrt{2}} = n(y)\sin\theta$$

$$\Rightarrow 1 - \frac{y}{L} = \frac{1}{\sin\theta}\frac{1}{\sqrt{2}}$$

$$\Rightarrow y = -L\left(\frac{1}{\sqrt{2}\sin\theta} - 1\right)$$

$$\Rightarrow dy = +\frac{L}{\sqrt{2}}\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

So,

$$\frac{d\theta}{dx} = +\frac{\sqrt{2}}{L}\frac{\sin^2\theta}{\cos\theta}\cot\theta = +\frac{\sqrt{2}}{L}\sin\theta$$

$$\Rightarrow \int_{\pi/4}^{\pi/2}\frac{d\theta}{\sin\theta} = \frac{\sqrt{2}}{L}\int_0^{\frac{x_{max}}{2}} dx \qquad (1)$$

Spring 2000 #2 (p 2 of 2)

From the hint we know that the LHS of eq (1) is

$$\int_{\pi/4}^{\pi/2} \frac{d\theta}{\sin\theta} = \frac{1}{2}\ln\left[\frac{1-\cos\theta}{1+\cos\theta}\right]\Bigg|_{\pi/4}^{\pi/2} = \frac{1}{2}\ln\left[\frac{1}{1}\right] - \frac{1}{2}\ln\left[\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\right]$$

So, substituting this in eq (1), we get

$$\frac{\sqrt{2}}{L}\frac{X_{max}}{2} = -\frac{1}{2}\ln\left[\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\right] = \frac{1}{2}\ln\left[\frac{1+\frac{\sqrt{2}}{2}}{1-\sqrt{2}/2}\right]$$

$$\therefore \boxed{X_{max} = \frac{L}{\sqrt{2}}\ln\left[\frac{2+\sqrt{2}}{2-\sqrt{2}}\right]}$$