Spring 2000 # 8

Consider an electron moving in crossed electric and magnetic fields, $\vec{E}=(0,E,0)$ $\vec{B}=(0,0,B)$ where E and B are constants.

a) Write down Schrödinger's equation

the Hamiltonian is $H=\frac{1}{2m}(\vec{p}-\frac{e}{c}\vec{A})^2+e\phi$

$\phi=-\int\vec{E}\cdot d\vec{l} = -Ey$

$\vec{B}=\nabla\times\vec{A} \Rightarrow B\hat{z}=\frac{d}{dx}A_y-\frac{d}{dy}A_x$ choose $A_x=-By$

$H=\frac{1}{2m}[(p_x+\frac{e}{c}By)^2+p_y^2+p_z^2]-eEy$

$$\boxed{\frac{1}{2m}[(p_x+\frac{e}{c}By)^2+p_y^2+p_z^2]-eEy = \mathcal{E}\Psi}$$ ($\mathcal{E}$ = energy)

b) The Schrödinger eq is separable in the right gauge. Reduce it to an ordinary differential equation.

→ Write Ψ as $\Psi(x,y,z)=X(x)Y(y)Z(z)$

the Schrödinger equation then becomes

$$\frac{1}{2m}\left(-\hbar^2\left(\frac{d^2}{dx^2}X\right)YZ-\frac{2ieBy\hbar}{c}\left(\frac{d}{dx}X\right)YZ+\left(\frac{e}{c}By\right)^2XYZ-\hbar^2\left(\frac{d^2}{dy^2}Y\right)XZ-\hbar^2\left(\frac{d^2}{dz^2}Z\right)XY\right.$$

$$-eEyXYZ=\mathcal{E}XYZ$$

dividing by XYZ and grouping terms yields

$$\left[\begin{array}{l}-\frac{\hbar^2}{2m}\frac{1}{X}\left(\frac{d^2}{dx^2}X\right)-\frac{ieBy\hbar}{cm}\frac{1}{X}\left(\frac{d}{dx}X\right)\\ -\frac{\hbar^2}{2m}\frac{1}{Y}\left(\frac{d^2}{dy^2}Y\right)+\left(\frac{e}{c}By\right)^2+eEy\\ -\frac{\hbar^2}{2m}\frac{1}{Z}\left(\frac{d^2}{dz^2}Z\right)\end{array}\right]=\mathcal{E}$$

Since these functions of X, Y, Z add up to a constant, each function must separately be constant.

One way for functions of x and z to be constant would be if $X(x)$ and $Z(z)$ were plane waves, so set

$X(x)=e^{ik_xx}$; $Z(z)=e^{ik_zz}$

Plug these solutions into Schrodinger's equation to obtain:

$$\boxed{\frac{1}{2m}\left[(+\hbar k_x+\frac{e}{c}By)^2+p_y^2+\hbar^2k_z^2\right]Y(y)-eEyY(y)=\mathcal{E}Y(y)} \Rightarrow$$

Schrödinger's eq:

$$\frac{1}{2m}\left(\hbar^2 k_x^2 + \frac{e^2 B^2 y^2}{c^2} + 2\hbar k_x \frac{eBy}{c} - 2meEy + p_y^2\right) Y(y) = \left(\mathcal{E} - \frac{\hbar^2 k_z^2}{2m}\right) Y(y)$$

Note that the Schrö eq can be written in the form of a harmonic oscillator: $\frac{1}{2m}p_y^2 + (ay - b)^2 + c$

Must complete the square for y.

$$\left(\frac{e^2B^2y^2}{2mc^2} + y\left(\frac{\hbar k_x eB}{mc} - eE\right) + \hbar^2 k_x^2\right) Y = \text{constant}$$

$$\text{constant} = \frac{p_y^2}{2m} + \mathcal{E} - \frac{\hbar^2 k_z^2}{2m}$$

$$\left(y^2 + y\,\frac{2mc^2}{e^2B^2}\left(\frac{\hbar k_x eB}{mc} - eE\right)\right) = \left(\frac{\text{constant}}{Y}\right)\left(\frac{2mc^2}{e^2B^2}\right) - \frac{2\hbar^2 k_x^2 mc^2}{e^2B^2}$$

$$\left(y + \underbrace{\frac{mc^2}{e^2B^2}\left(\frac{\hbar k_x eB}{mc} - eE\right)}_{\alpha}\right)^2 = \left(\frac{\text{constant}}{Y}\right)\left(\frac{2mc^2}{e^2B^2}\right) - \frac{2\hbar^2 k_x^2 mc^2}{e^2B^2} + \alpha^2$$

multiply by $Y\frac{e^2B^2}{2mc^2}$ to restore to proper form

$$\frac{e^2B^2}{2mc^2}\left(y + \frac{mc^2}{e^2B^2}\left(\frac{\hbar k_x eB}{mc} - eE\right)\right)^2 + \frac{p_y^2}{2m} = \mathcal{E}' Y$$

$$\left[\frac{p_y^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}y + \hbar k_x - \frac{2mcE}{B}\right)^2\right] Y = \mathcal{E}' Y$$

where $\mathcal{E}' = \mathcal{E} - \hbar^2 k_z^2 + \frac{4m^2c^2E^2}{B^2} - \frac{2mcE\hbar k_x}{B}$

Compare with Harmonic oscillator

$$\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2(y^2)\right] Y = \mathcal{E} Y \qquad \mathcal{E}_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

substitute $\omega = eB/mc$ to get:

$$\mathcal{E}' = \hbar\frac{eB}{mc}\left(n + \tfrac{1}{2}\right)$$

solving for $\mathcal{E}$ yields

$$\boxed{\mathcal{E} = \hbar\frac{eB}{mc}\left(n + \tfrac{1}{2}\right) + \hbar^2 k_z^2 - \frac{4m^2c^2E^2}{B^2} + \frac{2mcE\hbar k_x}{B}}$$