Consider an electron moving in crossed electric and magnetic fields, $\vec{E}=(0,E,0)$, $\vec{B}=(0,0,B)$, where E and B are constants.

a) Write down Schrödinger's equation (in any gauge).

- $H=\frac{1}{2m}(\vec{p}^{\,2}-q\vec{A})^2+q\phi$ SI

- $H=\frac{1}{2m}(\vec{p}^{\,2}-\frac{q\vec{A}}{c})^2+q\phi$ Gaussian

with $\vec{E}=-\vec{\nabla}\phi$ so $\phi=-Ey$

$\vec{B}=\vec{\nabla}\times\vec{A}$ so $\vec{A}=(-By,0,0)$

b) This equation is seperable in the right gauge. Reduce it to an ordinary differential equation.

$$H\psi=\frac{1}{2m}(-i\hbar\vec{\nabla}-q\vec{A})^2\psi+q\phi\psi$$

$$=\frac{1}{2m}\left(-\hbar^2\vec{\nabla}^2\psi+i\hbar q\vec{\nabla}\cdot(\vec{A}\psi)+i\hbar q\vec{A}\cdot(\vec{\nabla}\psi)+q^2\vec{A}^2\psi\right)+q\phi\psi$$

$$\vec{\nabla}\cdot(\vec{A}\psi)=\psi(\cancel{\vec{\nabla}\cdot\vec{A}})^{\to 0}+\vec{A}\cdot(\vec{\nabla}\psi)=\vec{A}\cdot(\vec{\nabla}\psi)$$

$$\vec{A}^2=B^2y^2$$

$$H\psi=\frac{1}{2m}\left(-\hbar^2\vec{\nabla}^2\psi+2i\hbar q\overbrace{\vec{A}\cdot(\vec{\nabla}\psi)}^{-By\frac{\partial}{\partial x}\psi}+q^2B^2y^2\psi\right)+q\phi\psi$$

$$=-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi+\frac{qBy}{m}\overbrace{(-i\hbar\frac{\partial}{\partial x}\psi)}^{p_x}+\frac{q^2B^2}{2m}y^2\psi-qEy\psi=$$

define: $\omega = \frac{qB}{m}$

so $H\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\omega^2 y^2\psi + (\omega p_x - qE)\,y\psi$

this can be separated into x, y, z

x: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E_x\psi \Rightarrow E_x = \frac{\hbar^2}{2m}k_x^2$

z: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi = E_z\psi \Rightarrow E_z = \frac{\hbar^2}{2m}k_z^2$

y: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m\omega^2 y^2\psi + \underbrace{(\omega p_x - qE)}_{\equiv C}\,y\psi = E_y\psi$

complete the squares:

$$\frac{1}{2}m\omega^2\Big(\underbrace{y + \frac{C}{m\omega^2}}_{\equiv y'}\Big)^2 = \frac{1}{2}m\omega^2 y^2 + Cy + \frac{C^2}{2m\omega^2}$$

so $\frac{1}{2}m\omega^2 y^2 + Cy = \frac{1}{2}m\omega^2 (y')^2 - \frac{C^2}{2m\omega^2}$

hence: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m\omega^2 (y')^2\psi - \frac{C^2\psi}{2m\omega^2} = E_y\psi$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m\omega^2 (y')^2\psi = E_y'\psi = \left(E_y + \frac{C^2}{2m\omega^2}\right)\psi$$

$E_y' = \hbar\omega(n+\tfrac{1}{2}) \Rightarrow E_y = \hbar\omega(n+\tfrac{1}{2}) - \frac{C^2}{2m\omega^2}$

so the total energy is: $E_{tot} = E_x + E_y + E_z$

$: \frac{\hbar^2}{2m}k_x^2 + \hbar\omega(n+\tfrac{1}{2}) - \frac{C^2}{2m\omega^2} + \frac{\hbar^2}{2m}k_z^2$

now $C^2 = (\omega p_x - qE)^2 = (\hbar\omega k_x - qE)^2 = \hbar^2\omega^2 k_x^2 - 2qE\hbar\omega k_x + q^2E^2$

so $$\frac{c^2}{2m\omega^2} = \frac{\hbar^2}{2m} k_x^2 - \frac{qE\hbar k_x}{m\omega} + \frac{q^2E^2}{2m\omega^2}$$

hence

$$E_{tot.} = \cancel{\frac{\hbar^2}{2m} k_x^2} + \hbar\omega(n+1/2) - \cancel{\frac{\hbar^2}{2m} k_x^2} + \frac{qE\hbar k_x}{m\omega} - \frac{q^2E^2}{2m\omega^2} + \frac{\hbar^2}{2m} k_z^2$$

$$= \hbar\omega(n+1/2) + \frac{\hbar^2}{2m} k_z^2 + \frac{qE\hbar k_x}{m\omega} - \frac{q^2E^2}{2m\omega^2}$$