QM S'00 #9

The proton in the H-atom is no longer a point charge, but a spherical shell of radius R ($R \ll a_0$).

a) Using perturbation theory what is the change in binding energy?

' Assume we are dealing with the ground state:

For the perturbation we want to replace the ~~point like~~ potential of the point like proton with that of a spherical shell of equal charge for $r<R$.

$$V(r)_{point} = \frac{1}{4\pi\epsilon_0}\frac{e}{r} \quad \text{(everywhere)}$$

$$V(r)_{shell} = \frac{1}{4\pi\epsilon_0}\frac{e}{r} \qquad r>R$$

$$\frac{1}{4\pi\epsilon_0}\frac{e}{R} \qquad r<R$$

So $$H' = \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{R}\right) \quad \text{for } r<R$$

$$= 0 \qquad r>R$$

$$E_0^{(1)} = \langle \psi_{100}^{(0)} | H' | \psi_{100}^{(0)} \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{e^{-r/a_0}}{(\pi a_0^3)^{1/2}} \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r}-\frac{1}{R}\right)\frac{e^{-r/a_0}}{(\pi a_0^3)^{1/2}}\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= \frac{e^2}{4\pi\epsilon_0}\frac{1}{\pi a_0^3}\underbrace{\int_0^{2\pi} d\phi}_{=2\pi}\ \underbrace{\int_0^{\pi}\sin\theta\, d\theta}_{=2}\int_0^{\infty} r^2\left(\frac{1}{r}-\frac{1}{R}\right)e^{-2r/a_0}\,dr$$

now the integral:

$$\int_0^\infty r^2\left(\frac{1}{r}-\frac{1}{R}\right)e^{-2r/a_0}\,dr = \int_0^R r^2\left(\frac{1}{r}-\frac{1}{R}\right)e^{-2r/a_0}\,dr$$

but $e^{-2r/a_0}\Big|_0^R = e^{-2R/a_0}$ and 1 but $R \ll a_0$ so $e^{-2R/a_0} \approx e^0 \approx 1$

so

$$\int_0^R r^2\left(\frac{1}{r}-\frac{1}{R}\right)e^{-2r/a_0}\,dr \approx \int_0^R r^2\left(\frac{1}{r}-\frac{1}{R}\right)dr = \int_0^R r\,dr - \frac{1}{R}\int_0^R r^2\,dr$$

$$= \frac{R^2}{2} - \frac{1}{3\cancel{R}}R^{\cancel{3}2} = \frac{1}{6}R^2$$

$$\Rightarrow \quad E_0^{(1)} = \frac{e^2}{\epsilon_0 \pi a_0^3}\cdot\frac{1}{6}R^2 = \boxed{\frac{e^2}{6\epsilon_0 \pi a_0^3}R^2}$$

b) The ground state energy is always either unchanged or decreased by perturbation.

c) $$\frac{(1.6\cdot 10^{-19}\,C)^2}{6(8.85\cdot 10^{-12}\frac{C^2}{Nm^2})\,\pi(0.5\cdot 10^{-10}\,m)^3}(10^{-15}\,m)^2 \approx \frac{(3/2\cdot 10^{-19})^2}{6\cdot 9\cdot 10^{-12}\cdot 3\cdot\frac{1}{8}\cdot\cancel{10^{-30}}}\cdot\cancel{10^{-30}} = \frac{9/4\cdot 10^{-38}}{\frac{3}{8}\cdot 6\cdot 9\cdot 10^{-12}} = \frac{1}{9}\cdot 10^{-26}\,J$$

$$= \frac{1}{9}\cdot 10^{-7}\,eV\cdot\frac{2}{3} = \boxed{\frac{2}{27}\cdot 10^{-7}\,eV}$$