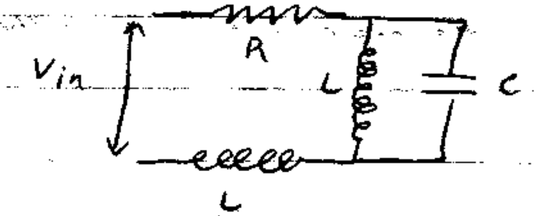
In the ac circuit shown below, the input voltage V_{in} and the circuit elements R, L, C are known quantities.



(a) Find the frequency $\omega = \omega_{res}$ at which the input impedance $Z_{in} = V_{in}/I_{in}$ is real.

$Z_R = R \; ; \; Z_L = i\omega L \; ; \; Z_C = -\frac{i}{\omega C}$

Need equivalent Z for $Z_L \parallel Z_C$ part

$$Z'_{eq} = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{i\omega L(-i/\omega C)}{i\omega L - \frac{i}{\omega C}} = \frac{L/C}{i(\omega L - \frac{1}{\omega C})} \Rightarrow \frac{-iL}{\omega LC - 1/\omega}$$

Then the total equivalent impedance is just

$$Z_{eq} = Z_R + Z'_{eq} + Z_L = R + \frac{-iL}{\omega LC - 1/\omega} + iL\omega = R + i\left[\omega L - \frac{L}{\omega LC - 1/\omega}\right]$$

We are told that at $\omega = \omega_{res}$ Z_{eq} is just real (i.e. $= R$), so

$$\omega_r L - \frac{L}{\omega_r LC - 1/\omega_r} = 0 \Rightarrow \omega_r \cancel{L} = \frac{\cancel{L}}{\omega_r LC - 1/\omega_r} \Rightarrow \omega_r^2 LC - 1 = 1 \Rightarrow \omega_r^2 = \frac{2}{LC}$$

so $\boxed{\omega_{res} = \omega_r = \sqrt{\frac{2}{LC}}}$

(b) Assuming L and C are ideal (i.e. non-dissipative), then the only loss is due to the resistor:

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \cos\phi$$

where V_0 & I_0 are the maximum values of voltage and current. ϕ is the angle between the voltage and current.

now

$$\phi = \arctan\left(\frac{X}{R}\right)$$

X is the reactance and R is the resistance in $Z = R + iX$

at $\omega = \omega_{res}$ $X = 0$ so $\phi = 0$ as well, hence

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \quad \text{or} \quad I_0 = \frac{V_0}{R} \quad \text{or} \quad V_0 = I_0 R$$

in terms of known quantities $\boxed{\langle P \rangle = \frac{1}{2} \frac{V_0^2}{R}}$

but for (d) we will actually use $\langle P \rangle = \frac{1}{2} I_0^2 R$

(c) At $\omega = \omega_{res}$, what is the stored energy, U_{stored}?

Solving part (d) first and reverse engineering $\boxed{U_{stored} = \frac{1}{2} L I^2}$ as for why that is as opposed to $\frac{1}{2} C V^2$ I don't know.

(d) Find the quality factor $Q = \omega_{res} \frac{U_{stored}}{P_{dissip}}$?

From the above two answers:

$$Q = \omega_{res} \frac{\cancel{\frac{1}{2}} L \cancel{I^2}}{\cancel{\frac{1}{2}} \cancel{I^2} R} = \frac{\omega_{res} L}{R}$$

As for the reverse engineering part, first solve the simpler series R, L, C circuit (see "Basic Electronics for Scientists", Brophy, p. 82)

$$Z_{eq} = Z_R + Z_L + Z_C = R + i\left[\omega L - 1/\omega C\right]$$

at ω_{res} (where $\omega_{res} = \frac{1}{\sqrt{LC}}$) $\quad Z_{eq} = R + i\left[\sqrt{\frac{L}{C}} - \sqrt{\frac{L}{C}}\right] = R$

(which is exactly what we have).

So we can write in general

$$I = \frac{V}{|Z|} = \frac{V}{|R + i(\omega L - 1/\omega C)|} = \frac{V}{R}\frac{1}{|1 + \frac{i}{R}(\omega L - 1/\omega C)|} \uparrow \frac{V}{R}\frac{1}{\left(1 + \frac{1}{R^2}(\omega L - \frac{1}{\omega C})^2\right)^{1/2}}$$

as $|\underset{\text{resistive}}{R} + i\underset{\text{reactive}}{X}| = \sqrt{R^2 + X^2}$

which can also be written as:

$$I = \frac{V}{R}\frac{1}{\left(1 + \underbrace{\left(\frac{\omega L}{R}\right)^2}_{\equiv Q^2 \text{ like}}\left[1 - \frac{1}{\omega^2 LC}\right]^2\right)^{1/2}}$$

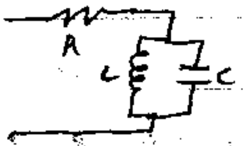
We can do a similar thing in our case:

$$I = \frac{V}{|Z|} = \frac{V}{|R + i(\omega L - \frac{L}{\omega LC - 1/\omega})|} = \frac{V}{R}\frac{1}{|1 + \frac{i}{R}(\omega L - \frac{L}{\omega LC - 1/\omega})|}$$

$$= \frac{V}{R}\frac{1}{\left(1 + \underbrace{\left(\frac{\omega L}{R}\right)^2}_{\equiv Q^2}\left[1 - \frac{1}{\omega^2 LC - 1}\right]^2\right)^{1/2}}$$

Check $\omega = \omega_{res}$: $\quad \frac{V}{R}\frac{1}{\left(1 + Q^2\left[1 - \frac{1}{\frac{2\cancel{LC}}{\cancel{LC}} - 1}\right]^2\right)^{1/2}} = \frac{V}{R}\frac{1}{\left(1 + Q^2\left[1 - \frac{1}{2-1}\right]^2\right)^{1/2}} = \frac{V}{R}$ ✓

Note: If the series inductance were removed



then this circuit would have an infinite impedance at ω_{res} and no current would flow! Amazing what a simple inductor in the right place can do.