The Hamiltonian for a system consisting of three distinguishable spin one particles is

$$H = A\left(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_3\cdot\vec{S}_1\right)$$

where $\vec{S}_i$ is the spin of the ith particle, and all the components of the spin of one particle commute with all the components of the spins of the other two.

(a) How many independent states are there?

$$\underset{\substack{\uparrow\\3}}{1} \otimes \underset{\substack{\uparrow\\3}}{1} \otimes \underset{\substack{\uparrow\\3}}{1} = 1\otimes(2\oplus 1\oplus 0) = 1\otimes 2\oplus 1\otimes 1\oplus 1\otimes 0$$

3 states

so $3\cdot 3\cdot 3 = 27$ states

$$= \underbrace{3\oplus 2\oplus 1}_{2\otimes 1}\oplus\underbrace{2\oplus 1\oplus 0}_{1\otimes 1}\oplus\underbrace{1}_{1\otimes 0}$$

$$= \underset{\substack{\uparrow\\7}}{3}\oplus\underset{\substack{\uparrow\\5}}{2}\oplus\underset{\substack{\uparrow\\5}}{2}\oplus\underset{\substack{\uparrow\\3}}{1}\oplus\underset{\uparrow}{1}\oplus\underset{\uparrow}{1}+\underset{\substack{\uparrow\\1}}{0} \text{ states}$$

or 27 states (just a check)

(b) what are the eigenvalues of H?

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + S_3 \rightarrow \vec{S}^2 = (\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_1\cdot\vec{S}_3 + \vec{S}_2\cdot\vec{S}_3)$$

or $\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_3\cdot\vec{S}_1 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 - \vec{S}_3^2\right)$

now $\vec{S}_i^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$ but as we are dealing with spin 1 particles $s=1 \Rightarrow \vec{S}_i^2 = 2\hbar^2$

Hence

$$H = \frac{A}{2}\left(\vec{S}^2 - 6\hbar^2\right)$$

Now when total s is $= 3$ $\quad H = \frac{A}{2}\left(\hbar^2 3(3+1) - 6\hbar^2\right) = 3A\hbar^2$

$= 2 \quad H = \frac{A}{2}\left(\hbar^2 2(2+1) - 6\hbar^2\right) = 0$

$= 1 \quad H = \frac{A}{2}\left(\hbar^2 1(1+1) - 6\hbar^2\right) = -2A\hbar^2$

$= 0 \quad H = \frac{A}{2}\left(\hbar^2 0(0+1) - 6\hbar^2\right) = -3A\hbar^2$

(c) What are the degeneracies of each energy level?

$3A\hbar^2$ $\quad$ 7× degeneracy

0 $\quad$ 10×

$-2A\hbar^2$ $\quad$ 9×

$-3A\hbar^2$ $\quad$ 1×