

Problem #2 Fall 2001.

$$H = A (\hat{S}_1 \cdot \hat{S}_2 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_3 \cdot \hat{S}_1)$$

a) $1 \otimes 1 \otimes 1$

$$\begin{array}{c|c} \begin{array}{cc} 111 & 100 \\ 110 & 010 \\ 101 & 100 \\ 011 & 000 \end{array} & \begin{array}{c} + \\ \\ \\ \end{array} \begin{array}{c|c} \begin{array}{ccc} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{array} & \begin{array}{c} + \\ \\ \\ \end{array} \begin{array}{c|c} \begin{array}{ccc} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{array} & \end{array} \\ \hline \begin{array}{c} \times 2 \\ \\ \\ \end{array} & \begin{array}{c} \text{not} \\ \text{multiplied} \\ \text{by } 2 \end{array} & \begin{array}{c} \times 2 \\ \\ \\ \end{array} & \begin{array}{c} \times 2 \\ \\ \\ \end{array} \end{array}$$

27 states = 3 particles with 3 states
 $= 3^3 = (\# \text{ of states})^{(\# \text{ of particles})}$

b) $|S_1 m_1, S_2 m_2, S_3 m_3\rangle = |m_1, m_2, m_3\rangle$

$$\hat{S}_1 \cdot \hat{S}_2 = \hat{S}_{1z} \hat{S}_{2z} + \frac{1}{2} (\hat{S}_{1+} \hat{S}_{2-} + \hat{S}_{1-} \hat{S}_{2+})$$

$$H = A \left(\hat{S}_{1z} \hat{S}_{2z} + \frac{1}{2} (\hat{S}_{1+} \hat{S}_{2-} + \hat{S}_{1-} \hat{S}_{2+}) + \hat{S}_{2z} \hat{S}_{3z} + \frac{1}{2} (\hat{S}_{2+} \hat{S}_{3-} + \hat{S}_{2-} \hat{S}_{3+}) + \hat{S}_{3z} \hat{S}_{1z} + \frac{1}{2} (\hat{S}_{3+} \hat{S}_{1-} + \hat{S}_{3-} \hat{S}_{1+}) \right)$$

$$\hat{S}^2 = (\hat{S}_1 + \hat{S}_2 + \hat{S}_3)(\hat{S}_1 + \hat{S}_2 + \hat{S}_3)$$

$$= \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 + 2\hat{S}_1 \cdot \hat{S}_2 + 2\hat{S}_1 \cdot \hat{S}_3 + 2\hat{S}_2 \cdot \hat{S}_3$$

$$H = \frac{A}{2} (\hat{S}^2 - (\hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2))$$

$$H|l\rangle = \frac{A}{2} \left\{ S(S+1)\hbar^2 - [S_1(S_1+1)\hbar^2 + S_2(S_2+1)\hbar^2 + S_3(S_3+1)\hbar^2] \right\} |l\rangle$$

$$E_3 = \frac{A}{2} \{ 12\hbar^2 - 6\hbar^2 \} = 3A\hbar^2$$

$$E_2 = \frac{A}{2} \{ 6\hbar^2 - 6\hbar^2 \} = 0$$

$$E_1 = \frac{A}{2} \{ 2\hbar^2 - 6\hbar^2 \} = -2A\hbar^2$$

$$E_0 = \frac{A}{2} \{ 0 - 6\hbar^2 \} = -3A\hbar^2$$

$$\begin{array}{l} (1,1,1), (-1,-1,-1) \\ \hline (1,0,1), (1,1,0) \\ (0,1,1), (-1,0,-1) \\ \hline (-1,-1,0), (0,-1,-1) \\ \hline (0,0,1), (0,1,0) \\ \hline (1,0,0), (-1,0,0) \\ (0,0,-1), (0,-1,0) \\ \hline (1,-1,1), (-1,1,1) \\ \hline (1,1,-1), (1,-1,-1) \\ (-1,1,-1), (-1,-1,1) \\ \hline (0,0,0), (0,1,-1) \\ (1,0,-1), (0,1,1) \\ (-1,0,1), (1,-1,0) \\ \hline (-1,1,0) \end{array}$$

0) $E_3 \rightarrow$ 2-fold degeneracy

$E_2 \rightarrow$ 6-fold degeneracy

$E_1 \rightarrow$ 12-fold degeneracy

$E_0 \rightarrow$ 7-fold degeneracy