Problem #6 Fall 2001

non-interacting conserved Bosons

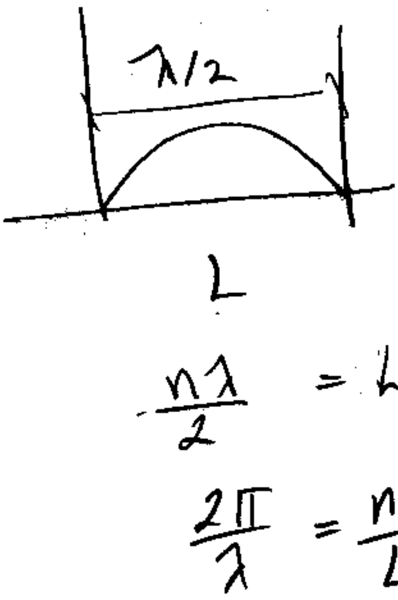
$$E(\vec{p}) = A|\vec{p}|^4 = \epsilon$$

a) $$N = \sum_\epsilon \frac{\bar{n}\, e^{-\beta\epsilon}}{\sum_\epsilon e^{-\beta\epsilon}}$$

for bosons $$\bar{n} = \frac{1}{e^{\beta\epsilon}-1}$$

$$N = \int \bar{n}(\epsilon) D(\epsilon)\, d\epsilon, \quad D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon}$$

$$\epsilon = A p^4 = A(\hbar K)^4$$
$$= A\left(\frac{\hbar n\pi}{L}\right)^4$$



$$\frac{L(\epsilon/A)^{1/4}}{\hbar\pi} = n = C\epsilon^{1/4}$$

$$\boxed{\frac{dn}{d\epsilon} = \frac{L}{\hbar\pi(A)^{1/4}}\frac{1}{4}\epsilon^{-3/4}}$$

same for all Dims

$$\boxed{\frac{dn}{d\epsilon} = \frac{C}{4}\epsilon^{-3/4}}$$

$$\frac{n\lambda}{2} = L$$

$$\frac{2\pi}{\lambda} = \frac{n\pi}{L} = K$$

$$p = \hbar K = \frac{hK}{2\pi}$$

$$p = \frac{h}{\lambda}$$

3D $\longrightarrow$ $\dfrac{dN}{dn} = \dfrac{d}{dn}\left(\dfrac{4}{3}\pi n^3\right) = 4\pi n^2$

$$N = \frac{c}{8}\int \frac{\epsilon^{-3/4}}{e^{\beta\epsilon}-1}\, 4\pi n^2 \, d\epsilon$$

$$n^2 = \frac{L^2\sqrt{\epsilon}}{\hbar^2\pi^2} = c^2\sqrt{\epsilon}$$

$$N = \frac{4\pi c^3}{8}\int_0^\infty \frac{\epsilon^{-1/4}}{e^{\beta\epsilon}-1}\, d\epsilon$$

$$= \frac{4\pi c^3}{8}\int_0^\infty e^{-\beta\epsilon}\left(\frac{\epsilon^{-1/4}}{1-e^{-\beta\epsilon}}\right) d\epsilon$$

$$= \frac{4\pi c^3}{8}\int_0^\infty e^{-\beta\epsilon}\, \epsilon^{-1/4} \sum_{l=0}^{\infty} \left(e^{-\beta\epsilon}\right)^l d\epsilon$$

$$= \frac{4\pi c^3}{8}\int_0^\infty \epsilon^{-1/4} \sum_{l=1}^{\infty} \left(e^{-\beta\epsilon}\right)^l d\epsilon$$

$$= \frac{4\pi c^3}{8}\sum_{l=1}^{\infty}\int_0^\infty \epsilon^{-1/4} e^{-\beta\epsilon l}\, d\epsilon$$

$$= \frac{4\pi c^3}{8}\sum_{l=1}^{\infty}\frac{(-1/4)!}{(\beta l)^{3/4}}$$

or any Dimension d

$$N = C_0 \int_0^\infty \frac{\epsilon^{-3/4}}{e^{\beta\epsilon}-1}\, n^{d-1}\, d\epsilon$$

$$n^{d-1} = B_0\, \epsilon^{(d-1)/4}$$

$$N = C_0 B_0 \int_0^\infty \frac{\epsilon^{\left(-3/4 + (d-1)/4\right)}\, d\epsilon}{e^{\beta\epsilon}-1}$$

$$= C_0 B_0 \int_0^\infty e^{-\beta\epsilon} \left(\frac{\epsilon^{\left(-3/4 + (d-1)/4\right)}}{1-e^{-\beta\epsilon}} \right) d\epsilon$$

$$= C_0 B_0 \sum_{l=1}^{\infty} \int_0^\infty e^{-\beta\epsilon l}\, \epsilon^{\left(-3/4 + (d-1)/4\right)}\, d\epsilon$$

$$= C_0 B_0 \underbrace{\sum_{l=1}^{\infty} \frac{\left(-3/4 + \frac{d-1}{4}\right)!}{(\beta l)^{d/4}}}_{\zeta\left(\frac{d}{4}\right)}$$

Therefore there are only physical solutions for $\frac{d}{4} > 1$. Therefore $d = 5$ is the lowest dimension for which a BE condensate will form.

b) $$\int f(|\vec{p}|)\, d^d p = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int f(p)\, p^{d-1}\, dp$$

$$N = \int \bar{n}(\epsilon) D(\epsilon)\, d\epsilon$$

$$= \int_0^\infty \frac{1}{e^{\beta\epsilon} - 1} \frac{dN}{dn} \frac{dn}{d\epsilon}\, d\epsilon$$

$$k = \frac{n\pi}{L} \qquad p = \hbar k = \frac{\hbar n \pi}{L}$$

$$\epsilon = A\left(\frac{\hbar n \pi}{L}\right)^q \qquad \boxed{n = \frac{L}{\hbar\pi A^{1/q}}\, \epsilon^{1/q}}$$

$$N = \int \bar{n}(\vec{p})\, d^d p = \int \bar{n}(\epsilon) D(\epsilon)\, d\epsilon$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \int \frac{n^{d-1}}{e^{\beta\epsilon} - 1}\, dn$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\frac{L}{\hbar\pi A^{1/q}}\right)^{d-1} \int \frac{\epsilon^{\frac{d-1}{q}}}{e^{\beta\epsilon} - 1} \left(\frac{L}{q\hbar\pi A^{1/q}}\right) \epsilon^{\frac{1-q}{q}}\, d\epsilon$$

$$\frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{L}{\hbar\pi A^{1/q}}\right)^d\frac{1}{q}\int_0^\infty\frac{\epsilon^{\frac{d-q}{q}}}{e^{\beta\epsilon}-1}\,d\epsilon$$

$$N=\frac{1}{2^d}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{L}{\hbar\pi A^{1/q}}\right)^d\frac{1}{q}\int_0^\infty\frac{\epsilon^{\frac{d-q}{q}}}{e^{\beta\epsilon}-1}\,d\epsilon$$

$$=C\int_0^\infty\frac{\epsilon^{\frac{d-q}{q}}}{e^{\beta\epsilon}-1}\,d\epsilon$$

$$=C\int_0^\infty\frac{e^{-\beta\epsilon}\,\epsilon^{\frac{d-q}{q}}}{1-e^{-\beta\epsilon}}\,d\epsilon$$

$$=C\int_0^\infty e^{-\beta\epsilon}\,\epsilon^{\frac{d-q}{q}}\sum_{l=0}^{\infty}e^{-\beta\epsilon l}\,d\epsilon$$

$$=C\sum_{l=1}\int_0^\infty e^{-\beta\epsilon l}\,\epsilon^{\frac{d-q}{q}}\,d\epsilon$$

$$=C\sum_{l=1}\frac{\left(\frac{d-q}{q}\right)!}{(\beta l)^{d/q}}$$

$$= C\,\Gamma(d/q)\,\frac{1}{\beta^{d/q}}\sum_{\ell=1}^{\infty}\frac{1}{\ell^{d/q}}$$

$$N = \frac{C\,\Gamma(d/q)}{\beta^{d/q}}\,\zeta(d/q)$$

for $d > q$

$$\boxed{T_c = \left(\frac{N}{C\,\Gamma(d/q)\,\zeta(d/q)}\right)^{q/d}\frac{1}{K_B}}$$