similar to Fall 2001 #6 ... more general

Calculation of conditions for BEC in an arbitrary number spatial dimensions

The conditions for the Bose-Einstein Condensate (BEC) to occur depend on the number of spatial dimensions under consideration, and the relationship of the energy to the momentum. Consider a general expression in which the energy ϵ is proportional to the momentum p, raised to some power q

$$\epsilon = Ap^q \tag{1}$$

This relation can be realized by comparison with a typical massive, non-interacting, non-relativistic boson for which $\epsilon = \frac{1}{2m}p^2$ or for a massless boson, ϵ = cp. In general, the evaluation of the condition for the BEC is done by calculation of the total number of particles through the relation

$$N = \int_0^\infty \bar{n}(\epsilon)dN = \int_0^\infty \frac{D(\epsilon)}{e^{\beta(\epsilon-\mu)}-1}d\epsilon \tag{2}$$

where D(ϵ) is the density of states, and $\beta=(k_bT)^{-1}$. For a particle in a box of side length L, the contained modes are quantized by the condition that the wave function vanish at the walls, Ψ(x,y,z=0)=Ψ(x,y,z=L)=0. Thus for each spatial dimension i, we have the condition

$$k_i = \frac{n_i\pi}{L} \tag{3}$$

From the deBroglie momentum relation we have $p_i=\hbar k_i$, so in terms of n_i we get $p_i=\hbar\frac{n_i\pi}{L}$. Inserting this into eq(1) we get

$$\epsilon = A\big(\hbar\frac{n\pi}{L}\big)^q \qquad \Rightarrow \qquad n = \frac{L}{\hbar\pi A^{1/q}}\epsilon^{1/q} \tag{4}$$

The density of states is evaluated by solving for the different pieces (I and II) of the derivative expansion

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = \underbrace{S_d(n)}_{(I)}\underbrace{\frac{dn}{d\epsilon}}_{(II)} \tag{5}$$

where (I) is the surface area of a d-dimentional sphere in n-space (n is the radius). This is given generally by

$$S_d(n) = \frac{2\pi^{d/2}n^{d-1}}{\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\big(\frac{L}{\hbar\pi A^{1/q}}\big)^{d-1}\epsilon^{\frac{d-1}{q}} \tag{6}$$

where we have inserted the expression for n from above. In lieu of memorizing this expression, it is sufficient to realize that the surface area term goes like n^{d-1}. From evaluation of term (II) using eq(4), we see that this term is independent of d. This is true for all such problems like this. Now solving completely for D(ϵ),

$$\begin{aligned} D(\epsilon) &= \Big\{\frac{2\pi^{d/2}}{\Gamma(d/2)}\Big(\frac{L}{\hbar\pi A^{1/q}}\Big)^{d-1}\epsilon^{\frac{d-1}{q}}\Big\}\Big\{\Big(\frac{L}{q\hbar\pi A^{1/q}}\Big)\epsilon^{\frac{1-q}{q}}\Big\} \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{q}\Big(\frac{L}{\hbar\pi}\Big)^d\frac{1}{A^{d/q}}\epsilon^{(d-q)/q} \end{aligned} \tag{7}$$

Note that we have an L^d term that gives us our volume element. Inserting D(ϵ) into eq(2) we have to remember to include a factor of $1/2^d$ since we are considering only positive values of n.

$$N = \underbrace{\frac{\pi^{d/2}}{2^{d-1}\Gamma(d/2)}\frac{1}{q}\Big(\frac{L}{\hbar\pi}\Big)^d\frac{1}{A^{d/q}}}_{C}\int_0^\infty \frac{\epsilon^{(d-q)/q}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon \tag{8}$$

Now for a couple tricks. Expand the denominator of the integrand in a power series of the form $\frac{1}{1-x} = \sum_{l=0}^\infty x^l$ valid when $|x| < 1$.

$$\begin{aligned} N &= C\int_0^\infty \frac{\epsilon^{(d-q)/q}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon \\ &= C\int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}}\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\epsilon^{(d-q)/q}d\epsilon \\ &= C\int_0^\infty e^{-\beta(\epsilon-\mu)}\Big(\sum_{l=0}^\infty e^{-l\beta(\epsilon-\mu)}\Big)\epsilon^{(d-q)/q}d\epsilon \\ &= C\int_0^\infty \Big(\sum_{l=1}^\infty e^{-l\beta(\epsilon-\mu)}\Big)\epsilon^{(d-q)/q}d\epsilon \\ &= C\sum_{l=1}^\infty\Big(\int_0^\infty e^{-l\beta(\epsilon-\mu)}\epsilon^{(d-q)/q}d\epsilon\Big) \\ &= C\sum_{l=1}^\infty e^{l\beta\mu}\Big(\int_0^\infty e^{-l\beta\epsilon}\epsilon^{(d-q)/q}d\epsilon\Big) \end{aligned} \tag{9}$$

The expansion has assumed that $e^{-l\beta(\epsilon-\mu)} < 1$ and thus $\epsilon > \mu$. The validity of this condition will become apparent shortly. We can now recognize the integral as of the form

$$\int_0^\infty e^{-ax}x^n = \frac{n!}{a^{n+1}} \tag{10}$$

Therefore we have

$$\begin{aligned} N &= C\sum_{l=1}^\infty e^{l\beta\mu}\left(\frac{(\frac{d-q}{q})!}{(l\beta)^{\frac{d-q}{q}+1}}\right) \\ &= C\frac{\Gamma(d/q)}{\beta^{d/q}}\underbrace{\sum_{l=1}^\infty\frac{e^{l\beta\mu}}{l^{d/q}}}_{L_{d/q}(e^{\beta\mu})} \qquad \Big\{\Gamma(d/q) = \big(\frac{d-q}{q}\big)!\Big\} \end{aligned} \tag{11}$$

The sum term in the underbrace is the Polylogarithmic function $L_{d/q}(e^{\beta\mu})$ also called a weighted Zeta function. For the BEC to occur $L_{d/q}(e^{\beta\mu})$ must approach it's maximum, finite value. This is given by $\mu = 0$, validating our expansion condition $\epsilon > \mu$=0 in that the energy of a single particle cannot be equal to zero. The sum then

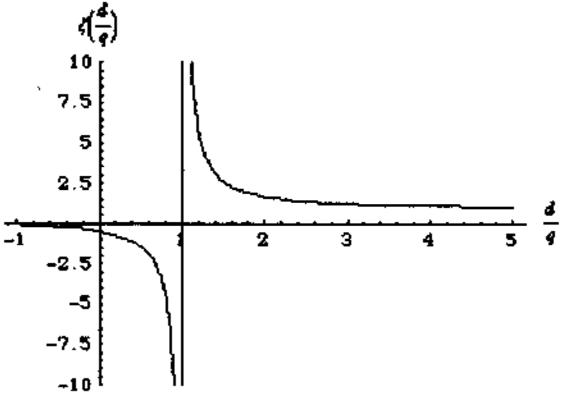

FIG. 1: The Riemann zeta function blows up at d/q → 1 and is negative for d/q < 1.

becomes the standard Riemann Zeta function $\zeta(x)$. The total number of particles is therefore

$$N = C\frac{\Gamma(d/q)}{\beta^{d/q}}\zeta(d/q) \tag{12}$$

A plot of $\zeta(d/q)$ is shown in FIG(1). When $d/q \leq 1$, the zeta function gives non-physical results in the form of infinite values or negative total particle numbers. Therefore the only physically tenable solutions for the BEC are obtained when the following relation is satisfied:

$$\boxed{d > q} \tag{13}$$

The expression for the BEC phase transition temperature can finally be obtained by rearranging eq(12).

$$T_c = \frac{1}{k_b}\left(\frac{N}{C\,\Gamma(d/q)\zeta(d/q)}\right)^{q/d} \tag{14}$$

In summary, the relationship between the momentum exponent q and the dimensionality d of the space determines whether the BEC will occur for the particle-in-a-box model. Evaluation of the integral yields the Riemann seta function for the BEC condition μ=0, which gives physical solutions *only if* $d > q$.