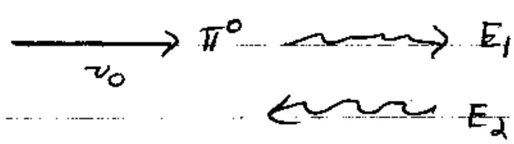
A π^0 of velocity v_0 decays in flight into two photons $\pi^0 \to 2\gamma$. Compute the minimum and maximum values of the energies of the produced photons as a function of v_0.

Three ways of doing it:

(1) If you can accept that the maximum energy comes from the photon traveling in the same direction as the π^0 and the minimum energy when traveling in the exact opposite direction:



then from energy conservation & momentum conservation

$$E_{\pi^0} = E_1 + E_2 \quad ; \quad |\vec{P}_{\pi^0}| = |\vec{p}_1| - |\vec{p}_2| = E_1 - E_2 \quad \text{with } c = 1$$

combining $\quad E_{\pi^0} + |\vec{P}_{\pi^0}| = 2E_1$

so $\quad$ as $E = \gamma m \, ; |p| = \gamma\beta m$

$$E_1 = \frac{1}{2}\left(E_{\pi^0} + |\vec{P}_{\pi^0}|\right) = \frac{1}{2}\left(\gamma m_{\pi^0} + \gamma\beta m_{\pi^0}\right)$$

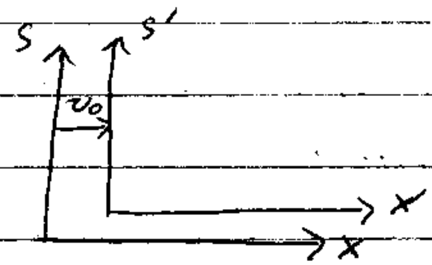
$$= \frac{\gamma m_{\pi^0}}{2}(1+\beta) \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

and the minimum energy would then be:

$$E_2 = \frac{\gamma m_{\pi^0}}{2}(1-\beta)$$

(2) Can be done via a Lorentz transformation from the rest frame of the π^0:

Just like
$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta c t')$$
$$y = y'$$
$$z = z'$$



a similar thing can be done with the energy-momentum four vector:

$$E/c = \gamma(E'/c + \beta p_x') = \gamma mc \Rightarrow E = \gamma mc^2$$
$$p_x = \gamma(p_x' + \beta E'/c) = \gamma\beta mc \rightarrow p_x = \gamma\beta mc^2$$

now in S' (π^0 rest frame) from momentum conservation $E_1' = \frac{M}{2} = p_{1,x}'$ (M: π^0)

$E_2' = \frac{M}{2} = -p_{2,x}'$. Then going into S (moving at v_0):

$$E_1 = \gamma(E_1' + \beta p_1') = \frac{M}{2}\gamma(1+\beta)$$
$$E_2 = \gamma(E_2' + \beta p_2') = \frac{M}{2}\gamma(1-\beta)$$

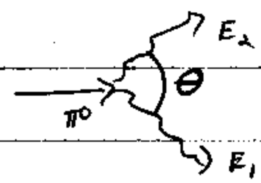
(3) $P = p_1 + p_2$ (4-vectors)

$$P^2 = M^2 = \cancel{p_1^2}^{\,0} + \cancel{p_2^2}^{\,0} + 2p_1 \cdot p_2 = 2[E_1E_2 \overset{\text{depends on metric used}}{-} \underbrace{2\vec{p}_1 \cdot \vec{p}_2}_{E_1E_2\cos\theta}]$$

as $m_\gamma = 0$

so $M^2 = 2E_1E_2(1-\cos\theta) \Rightarrow E_1E_2 = \frac{M^2}{2(1-\cos\theta)}$ $\quad\theta = 0 \quad E_1E_2 = \infty$

$\theta = \pi \quad E_1E_2 = \frac{M^2}{2}$



from energy conservation

$$E = E_1 + E_2 \qquad E_2 = \frac{1}{E_1}\frac{M^2}{2(1-\cos\theta)}$$
$$= E_1 + \frac{1}{E_1}\frac{M^2}{2(1-\cos\theta)} \qquad ax^2 + bx + c = 0 \qquad x = \frac{-b}{2a} \pm \frac{1}{2}\sqrt{b^2 - 4ac}$$

$$\Rightarrow E_1E = E_1^2 + \frac{M^2}{2(1-\cos\theta)} \Rightarrow E_1^2 - EE_1 + \frac{M^2}{2(1-\cos\theta)} = 0$$

$$\Rightarrow E_1 = -\frac{(-E)}{2} \pm \frac{1}{2}\sqrt{E^2 - \frac{4M^2}{2(1-\cos\theta)}} = \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - \frac{M^2}{2(1-\cos\theta)}} = \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - \frac{M^2}{4}}$$

smallest value $\theta = \pi$

So $E_1 = \frac{E}{2} \pm \frac{1}{2} \underbrace{\sqrt{E^2 - m^2}}_{|\vec{p}|} = \frac{E}{2} \pm \frac{1}{2} |\vec{p}|$

$= \frac{\gamma m_{\pi^0}}{2} \pm \frac{1}{2} \gamma \beta m_{\pi^0} = \frac{\gamma m_{\pi^0}}{2} (1 \pm \beta)$

hence the maximum energy is $\frac{\gamma m_{\pi^0}}{2}(1+\beta)$ and minimum $\frac{\gamma m_{\pi^0}}{2}(1-\beta)$