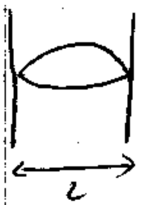
Stat. Mech. S'04 #6, S'01 # 13

For relativistic bosons

$$E = |\vec{p}|c$$

a) First we need the density of states $D(E)$ for 3-D:



$$\frac{L}{\lambda/2} = n \Rightarrow \frac{2L}{n} = \lambda; \quad p = \frac{h}{\lambda} = \frac{h}{2L}n$$

$$E = c|\vec{p}| = c\left(p_x^2 + p_y^2 + p_z^2\right)^{1/2} = \frac{hc}{2L}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}$$

$$= \frac{hc}{2L}n \Rightarrow n = \frac{2L}{hc}E \Rightarrow dn = \frac{2L}{hc}dE$$

$$\frac{(2s+1)}{8}\int_0^\infty 4\pi n^2 dn = \frac{(2s+1)}{8}\int 4\pi\left(\frac{2L}{hc}\right)^3 E^2 dE = \int_0^\infty \underbrace{\frac{(2s+1)4\pi \overset{L^3}{V}}{(hc)^3}E^2}_{D(E)} dE$$

$$D(E) = \frac{(2s+1)4\pi V}{(hc)^3}E^2$$

The condition for BEC is determined by the boson temperature T_B, which can be derived followingly:

$$\int_0^\infty \frac{1}{e^{E/kT}-1}D(E)dE = N \quad \text{for } T = T_B$$

$$\Rightarrow \frac{(2s+1)4\pi V}{(hc)^3}\int_0^\infty \frac{E^2}{e^{E/kT}-1}dE = \frac{(2s+1)4\pi V(kT)^3}{(hc)^3}\underbrace{\int_0^\infty \frac{x^2}{e^x-1}dx}_{=2.404}$$

$$x = E/kT \Rightarrow E = kTx \Rightarrow dE = kTdx$$

hence

$$\frac{(2s+1)\,4\pi V}{(hc)^3}(kT_B)^3\, 2.404 = N$$

$$\Rightarrow \quad (kT_B)^3 = \frac{N}{V}\,\frac{(hc)^3}{(2s+1)4\pi\cdot 2.404}$$

$$\Rightarrow \quad \boxed{T_B = \left(\frac{N}{k^3 V}\,\frac{(hc)^3}{(2s+1)4\pi\cdot 2.404}\right)^{1/3}}$$

b) yes it does occur - just derive $D(E)$ for 2-D case and repeat above steps:

$$E = \frac{hc}{2L}\left(n_x^2 + n_y^2\right) = \frac{hc}{2L}\,n \Rightarrow \quad n = \frac{2L}{hc}E \Rightarrow dn = \frac{2L}{hc}\,dE$$

$$\frac{(2s+1)}{4}\int_0^\infty 2\pi n\,dn = \frac{(2s+1)}{4}\int_0^\infty 2\pi\left(\frac{2L}{hc}\right)E\,dE = \int_0^\infty \underbrace{\frac{(2s+1)\,2\pi A}{(hc)^2}}_{}\,E\,dE$$

L^2

$$D(E) = \frac{(2s+1)\,2\pi A}{(hc)^2}\,E$$

$$\int_0^\infty \frac{(2s+1)\,2\pi A}{(hc)^2}\,\frac{E}{e^{E/kT}-1}\,dE = \frac{(2s+1)\,2\pi A}{(hc)^2}(kT)^2\underbrace{\int_0^\infty \frac{x}{e^x-1}\,dx}_{\frac{\pi^2}{6}} = N$$

$$x \equiv E/kT \Rightarrow \quad E = kTx \Rightarrow dE = kT\,dx$$

so

$$(kT_B)^2 = \frac{N}{A}\,\frac{3(2s+1)}{\pi^3(hc)^2} \quad \Rightarrow \quad \boxed{T_B = \left(\frac{N}{k^2 A}\,\frac{3(2s+1)}{\pi^3(hc)^2}\right)^{1/2}}$$

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c) BEC does not occur in 1-D case.

$$E = c|\vec{p}| = \frac{hc}{2L}\,n \Rightarrow n = \frac{2L}{hc}E \Rightarrow dn = \frac{2L}{hc}\,dE$$

$$\frac{(2s+1)}{2}\int_0^\infty dn = \int_0^\infty \underbrace{\frac{(2s+1)}{\cancel{2}}\,\frac{\cancel{2}L}{hc}}_{D(E) = \frac{(2s+1)L}{hc}}\,dE$$

$$\int_0^\infty \frac{(2s+1)L}{hc}\,\frac{dE}{e^{-E/kT}-1} = \frac{(2s+1)L}{hc}\,kT\underbrace{\int_0^\infty \frac{dx}{e^x-1}}_{\text{undefined}} = \text{undefined}.$$

$$x \equiv E/kT \Rightarrow dE = kT\,dx$$