Spr. 2001 #1:

$\vec{S}_1, \vec{S}_2, \vec{S}_3$ all spin $\frac{1}{2}$ $\quad P_{12}|\sigma_1,\sigma_2,\sigma_3\rangle = |\sigma_2,\sigma_1,\sigma_3\rangle$

$P_{123}|\sigma_1,\sigma_2,\sigma_3\rangle = |\sigma_2,\sigma_3,\sigma_1\rangle$

$\sigma_{1,2,3} = \pm\frac{1}{2}$ are the eigenvalues of S_1^z, S_2^z, S_3^z

(a) express P_{12} in terms of S_1 + S_2

recall the old trick $S^2 = S_1^2 + 2S_1\cdot S_2 + S_2^2$

$2\vec{S}_1\cdot\vec{S}_2 = 2S_1^z S_2^z + S_1^+ S_2^- + S_1^- S_2^+$

the effect of this operator on $|\sigma_1,\sigma_2,\sigma_3\rangle$ is

(I) $2\vec{S}_1\cdot\vec{S}_2|\sigma_1,\sigma_2,\sigma_3\rangle = 2S_1^z S_2^z|\sigma_1,\sigma_2,\sigma_3\rangle = 2S_1^z S_2^z|\sigma_2,\sigma_1,\sigma_3\rangle = \frac{1}{2}|\sigma_2,\sigma_1,\sigma_3\rangle$, if $\sigma_1=\sigma_2$

(II) $2\vec{S}_1\cdot\vec{S}_2|\sigma_1,\sigma_2,\sigma_3\rangle = -\frac{1}{2}|\sigma_1,\sigma_2,\sigma_3\rangle + |\sigma_2,\sigma_1,\sigma_3\rangle$ if $\sigma_1 \neq \sigma_2$

in (I), can write $\frac{1}{2}|\sigma_2,\sigma_1,\sigma_3\rangle = |\sigma_2,\sigma_1,\sigma_3\rangle - \frac{1}{2}|\sigma_1,\sigma_2,\sigma_3\rangle$

so the expression in (II) always works

then have $|\sigma_2,\sigma_1,\sigma_3\rangle = P_{12}|\sigma_1,\sigma_2,\sigma_3\rangle$, so

$2\vec{S}_1\cdot\vec{S}_2 = -\frac{1}{2} + P_{12} \Rightarrow \boxed{P_{12} = \frac{1}{2} + 2\vec{S}_1\cdot\vec{S}_2}$

(b) express P_{123} as S_1, S_2, + S_3

write $P_{123} = P_{12}P_{13} = \left(\frac{1}{2} + 2\vec{S}_1\cdot\vec{S}_2\right)\left(\frac{1}{2} + 2\vec{S}_1\cdot\vec{S}_3\right) =$

$\frac{1}{4} + \vec{S}_1\cdot\vec{S}_3 + \vec{S}_1\cdot\vec{S}_2 + 4(\vec{S}_1\cdot\vec{S}_2)(\vec{S}_1\cdot\vec{S}_3)$