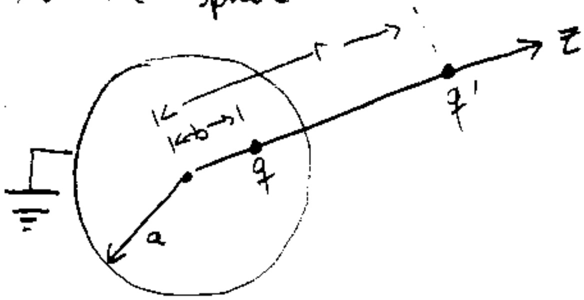
A point charge q is inside a hollow, grounded, conducting sphere of inner radius a. Use the method of images to find

(a) the potential inside the sphere



save yourself much time and memorize the following for image charge sphere problems (works whether charge is inside sphere or outside).

$$q' = -q\frac{a}{b} \quad ; \quad r = \frac{a^2}{b}$$

So, the potential inside sphere is

$$\Phi(r,\theta) = \frac{q}{\sqrt{x^2+y^2+(z-b)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z-r)^2}}$$

$$= \frac{q}{\sqrt{x^2+y^2+(z-b)^2}} - \frac{q\,(a/b)}{\sqrt{x^2+y^2+(z-\frac{a^2}{b})^2}}$$

note: the convention is to write

$$x^2+y^2+(z-\alpha)^2 = \underbrace{x^2+y^2+z^2}_{r^2} - 2z\cdot\alpha + \alpha^2 = r^2+\alpha^2-2z\alpha\cos\theta$$

So, we have

$$\boxed{\Phi(r,\theta) = \frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{q\,(a/b)}{\sqrt{r^2+\left(\frac{a^2}{b}\right)^2-2r\left(\frac{a^2}{b}\right)\cos\theta}}}$$

(b) the induced surface-charge density at the point on the sphere nearest to q.

$$\sigma = -\frac{1}{4\pi}\frac{\partial \Phi}{\partial r}\Bigg|_{r=a}$$

so,

$$-4\pi\sigma = \frac{-q(2r-2b\cos\theta)}{2(r^2+b^2-2rb\cos\theta)^{3/2}}\Bigg|_{r=a} - \frac{q(a/b)\left(2r-2\left(\frac{a^2}{b}\right)\cos\theta\right)}{2\left[r^2+\left(\frac{a^2}{b}\right)^2-2r\left(\frac{a^2}{b}\right)\cos\theta\right]^{3/2}}\Bigg|_{r=a}$$

$\therefore$

$$\boxed{\sigma = \frac{q}{4\pi}\left\{\frac{a-2b\cos\theta}{[a^2+b^2-2ab\cos\theta]^{3/2}} + \frac{a^2-a^3\cos\theta}{b\left[a^2+\left(\frac{a^2}{b}\right)^2-2\left(\frac{a^3}{b}\right)\cos\theta\right]^{3/2}}\right\}}$$

(c) the magnitude and direction of the force acting on q

note: if the question asked for force acting on surface of sphere it would be

$$F_z = 2\pi\int\sigma^2\,\hat{z}\cdot d\vec{a}$$

but it does not. So, we simply have (Griffiths' eq 3.18)

$$\vec{F} = \frac{qq'}{(r-b)^2}$$

$\therefore$

$$\boxed{F = \frac{-q^2(a/b)}{\left[\frac{a^2}{b}-b\right]^2} = \frac{-q^2ab}{(a^2-b^2)^2}}$$

Force must be in the $+\hat{z}$ direction

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

The answer to both questions is that there will be no change in the solution. Let's show this separately.

(i) if sphere is kept at a fixed potential V.

So, we have

$$\Phi' = \Phi + V \quad \text{, where } V \text{ is constant.}$$

$$\Rightarrow \sigma' = \frac{-1}{4\pi} \left.\frac{\partial \Phi'}{\partial r}\right|_{r=a} = \frac{-1}{4\pi}\left[\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} + \cancelto{0}{\left.\frac{\partial V}{\partial r}\right|_{r=a}} \right] = \sigma \checkmark$$

&

$$\vec{E}' = -\nabla\Phi' = -\nabla\Phi - \cancelto{0}{\nabla V} = \vec{E} \Rightarrow \vec{F}' = q\vec{E}' = q\vec{E} = \vec{F} \checkmark$$

(ii) if the sphere has a total charge Q on its inner and outer surfaces.

since Q is also constant wrt r, θ, ϕ, we get the same result as above, that is

$$\Phi' = \Phi + \frac{Q+q}{R} \quad \text{, where } R \text{ is outer radius}$$

since the second term is constant wrt r, θ, ϕ, no change will occur in σ and $\vec{F}$.

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(note: this is Jackson problem 2.2!)

In this case, we can fix an image charge somewhere outside the conducting shell. This image charge will function to hold the potential everywhere on the spherical surface at zero potential; that is, the image charge takes the place of the charge density that's present on the spherical shell.

Note that this problem is entirely equivalent to the one exhibited in Jackson section 2.2 except that the image charge needs to be at a distance *greater* than a. Let d denote charge q's radial distance from the center of the sphere; then you should find that the image charge's magnitude and distance are:

$$q_{im} = -\frac{qa}{d} \qquad |\vec{x}_{im}| = \frac{a^2}{d}$$

Note that $(a^2/d) > a$ since $d < a$. So the image charge is indeed outside the conducting sphere. The potential inside the sphere then is given by the following

$$\Phi_{in}(\vec{x}) = \frac{q}{|\vec{x} - d\hat{x}_q|} - \frac{qa}{d\left|\vec{x} - \frac{a^2}{d}\hat{x}_q\right|}$$

where $\hat{x}_q$ is the direction vector from the origin (center of sphere) to charge q's location.

(part b) Find the induced surface-charge density.

Just use the relation $\sigma = -(1/4\pi)(\partial\Phi/\partial x)_{x=a}$. After differentiation, algebra, and evaluation of the result at $x = a$, you should find

$$\sigma = -\frac{q}{4\pi a^2}\left(\frac{d}{a}\right)^2\left[\frac{1-(a/d)^2}{(1+(d/a)^2-2(d/a)\cos\gamma)^{3/2}}\right]$$

where γ is the angle between $\hat{x}_q$ and the position vector to some arbitrary field point $\vec{x}$.

(part c) Find the magnitude and direction of the force acting on q.

Since we're dealing with a point charge, we can use Coulomb's law:

$$\vec{F} = q\left(-\vec{\nabla}\Phi_{in}\right) = \frac{q^2}{d^2}\left(\frac{d}{a}\right)^3\left[1-\frac{d^2}{a^2}\right]^{-2}$$

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(part d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

If there's a fixed potential V on the surface, then we'de need to account for this potential inside the sphere by using a point charge of magnitude Va located at the center of the sphere. This will produce an extra piece in the potential given by

$$\Phi_{extra} = \frac{Va}{|\vec{x}|}$$

The total potential inside the sphere now will have a singularity at the origin. On the other hand, if there's a total charge Q on the surface of the condcuting sphere, there'll be no change in the potential expression for the interior of the sphere. This extra charge Q will be distributed uniformly on the sphere's surface; the isotropic nature of this surface charge density ensures that the interior E-field inside the sphere due to Q is zero (as you can readily verify using Gauss' law) and so the potential due to Q everywhere inside the sphere is an arbitrary constant which can be set to zero.