A Stern-Gerlach apparatus is adjusted so that the z-component of the spin of an electron (spin 1/2) transmitted through it is $-\hbar/2$. A uniform magnetic field in the x-direction is then switched on at time t = 0.

(a) what are the probabilities associated with finding the different allowed values of the z-component of the spin after time T?

an electron in an external magnetic field as the Hamiltonian (due to the B-field) of

$$H = -\vec{\mu}\cdot\vec{B} = \frac{g\mu_0}{2}\vec{B}\cdot\vec{S}$$

for our case $\vec{B} = B_0\hat{x}$, so, we have

$$H = \frac{g\mu_0 B_0}{2}\sigma_x = \frac{g\mu_0 B_0}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

this is the Hamiltonian at $t \geq 0$. Before that, we are told the eigenvalue of the electron is $-\hbar/2$ (for the z-component). This eigenvalue corresponds to the eigenstate

$$|\psi(t=0)\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we want to write the $|\psi(t=0)\rangle$ in terms of the eigenstates of H. That is, first let's find the eigenstates of H. (set $\hbar = 1$)

eigenvalues: $\det\begin{pmatrix} -\lambda & \frac{g\mu_0 B_0}{2} \\ \frac{g\mu_0 B_0}{2} & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \mu_0^2 B_0^2 = 0 \quad \therefore \lambda = \pm\mu_0 B_0$

where $g \equiv 2$ (since electron)

eigenstates: • $|\lambda = +\mu_0 B_0\rangle$: $\begin{pmatrix} -\mu_0 B_0 & \mu_0 B_0 \\ \mu_0 B_0 & -\mu_0 B_0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow -\mu_0 B_0 \phi_1 + \mu_0 B_0 \phi_2 = 0$

$\Rightarrow \phi_1 = \phi_2$

$$\Rightarrow |\lambda = +\mu_0 B_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

• $|\lambda = -\mu_0 B_0\rangle$; $\begin{pmatrix} \mu_0 B_0 & \mu_0 B_0 \\ \mu_0 B_0 & \mu_0 B_0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow -\phi_1 = \phi_2$

$$\Rightarrow |\lambda = -\mu_0 B_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So,

$$|\psi(t=0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|\lambda = \mu_0 B_0\rangle - \frac{1}{\sqrt{2}}|\lambda = -\mu_0 B_0\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +1 \end{pmatrix}\right] - \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now, apply the time evolution operator.

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{-i\mu_0 B_0 t}|\lambda = \mu_0 B_0\rangle - \frac{1}{\sqrt{2}} e^{i\mu_0 B_0 t}|\lambda = -\mu_0 B_0\rangle$$

Now, we want to write $|\psi(t)\rangle$ in terms of the eigenstates of S_z. So, we have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\mu_0 B_0 t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} e^{i\mu_0 B_0 t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{-1}{2}\begin{pmatrix} -e^{-i\mu_0 B_0 t} + e^{i\mu_0 B_0 t} \\ -e^{-i\mu_0 B_0 t} - e^{i\mu_0 B_0 t} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2i\sin(\mu_0 B_0 t) \\ -2\cos(\mu_0 B_0 t) \end{pmatrix}$$

$$= -\frac{1}{2} 2i\sin(\mu_0 B_0 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{2}\cos(\mu_0 B_0 t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \boxed{|\psi(t)\rangle = -i\sin(\mu_0 B_0 t)|\uparrow\rangle + \cos(\mu_0 B_0 t)|\downarrow\rangle}$$

Thus,

$$\boxed{\begin{aligned} P_{z\uparrow} &= |\langle\uparrow|\psi(t)\rangle|^2 = \sin^2(\mu_0 B_0 t) \\ P_{z\downarrow} &= |\langle\downarrow|\psi(t)\rangle|^2 = \cos^2(\mu_0 B_0 t) \end{aligned}}$$

$$P_z = \sin^2(\ ) + \cos^2(\ ) = 1 \ \checkmark$$

(b) what are the probabilities associated with finding the different allowed values of the x-component of the spin after time T?

From eq (1) we have

$$\boxed{\begin{aligned} P_{\cancel{x}\uparrow} &= |\langle\uparrow|\psi(t)\rangle|^2 = \frac{1}{2} \\ P_{\cancel{x}\downarrow} &= |\langle\downarrow|\psi(t)\rangle|^2 = \frac{1}{2} \end{aligned}}$$

$$P_x = \frac{1}{2} + \frac{1}{2} = 1 \ \checkmark$$