The Hamiltonian for a spinless charged particle in a magnetic field is:

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2$$

where the magnetic field B is related to the vector potential $\vec{A}$ by $\vec{B} = \vec{\nabla}\times\vec{A}$. Here, e is the charge of the particle, m the mass, c the velocity of light and $\vec{p} = (p_x, p_y, p_z)$ is the momentum of the particle. Let $\vec{A} = -B_0 y\hat{x}$, corresponding to the magnetic field $\vec{B} = B_0\hat{z}$

(a) Find the energy levels of the particle.

(b) Would the energy levels change if we chose $\vec{A}$ to be $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$? Give reasons for your answer.

(a) $H\psi = \frac{1}{2m}\left[\vec{p}^2\psi - \frac{e}{c}\vec{p}\cdot(\vec{A}\psi) - \frac{e}{c}\vec{A}\cdot(\vec{p}\psi) + \frac{e^2}{c^2}\vec{A}^2\psi\right]$

$\vec{p} = -i\hbar\vec{\nabla}$; $\vec{A} = -B_0 y\hat{x} \Rightarrow \vec{A}^2 = B_0^2 y^2$

also $\vec{p}\cdot(\vec{A}\psi) = -i\hbar\vec{\nabla}\cdot(\vec{A}\psi) = -i\hbar\left[\psi\underbrace{(\vec{\nabla}\cdot\vec{A})}_{0} + \vec{A}\cdot(\vec{\nabla}\psi)\right] = \vec{A}\cdot(\vec{p}\psi)$

and using the dot product: $\vec{A}\cdot(\vec{p}\psi) = -B_0 y p_x\psi$

so

$$H\psi = \frac{1}{2m}\left[\vec{p}^2\psi + \frac{2eB}{c}yp_x\psi + \frac{e^2}{c^2}\vec{A}^2\psi\right]$$

$$= \frac{p^2}{2m}\psi + \frac{eB}{mc}yp_x\psi + \frac{1}{2}\frac{e^2B_0^2}{mc^2}y^2\psi$$

define $\omega = \frac{eB}{mc}$

$$= \frac{p^2}{2m}\psi + \frac{1}{2}m\omega^2y^2\psi + \omega y p_x\psi$$

as $[p_x, H] = 0 = [p_z, H]$ one possible form of $\psi = e^{ik_x x}Y(y)e^{ik_z z}$

as x: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E_x\psi \Rightarrow E_x = \frac{\hbar^2}{2m}k_x^2$

z: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi = E_z\psi \Rightarrow E_z = \frac{\hbar^2}{2m}k_z^2$

y: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m\omega^2 y^2\psi + \underbrace{\omega y p_x\psi}_{\omega y(-i\hbar)(ik_x)\psi = \omega\hbar k_x y} = E_y\psi$

completing the square:

$$\frac{1}{2}m\omega^2\Big[\underbrace{y + \frac{\hbar k_x}{m\omega}}_{=y'}\Big]^2 = \frac{1}{2}m\omega^2 y^2 + \omega\hbar k_x y + \frac{\hbar^2 k_x^2}{2m}$$

so $\frac{1}{2}m\omega^2 y^2 + \omega\hbar k_x y = \frac{1}{2}m\omega^2(y')^2 - \frac{\hbar^2 k_x^2}{2m}$

$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y'^2}\psi + \frac{1}{2}m\omega^2(y')^2\psi = E'_y\psi = \left[E_y + \frac{\hbar^2 k_x^2}{2m}\right]\psi$

$E'_y = (n_y+1/2)\hbar\omega \Rightarrow E_y = (n_y+1/2)\hbar\omega - \frac{\hbar^2 k_x^2}{2m}$

so $E = E_x + E_y + E_z = \cancel{\frac{\hbar^2}{2m}k_x^2} + (n_y+1/2)\hbar\omega \cancel{- \frac{\hbar^2}{2m}k_x^2} + \frac{\hbar^2}{2m}k_z^2$

$$= (n_y+1/2)\hbar\omega + \frac{\hbar^2}{2m}k_z^2$$

(b) As energy is a measurable observable it can't depend upon the gauge you choose - hence the energy levels will not change.

A more detailed look:

$\vec{A} = -\frac{B_0}{2}y\hat{x} + \frac{B_0}{2}x\hat{y} \Rightarrow A^2 = \frac{B_0^2}{4}(x^2+y^2)$

$H\psi = \frac{1}{2m}\Big[\vec{p}^{\,2}\psi + \underbrace{\frac{2e}{c}\vec{A}\cdot(\vec{p}\psi)}_{\frac{2e}{c}\left[-\frac{B_0}{2}y p_x\psi + \frac{B_0}{2}x p_y\psi\right]} + \frac{e^2B_0^2}{c^2 4}(x^2+y^2)\psi\Big]$

$$H\psi = \frac{p^2}{2m}\psi + \frac{eB_0}{2mc}\underbrace{(xp_y - yp_x)}_{L_z}\psi + \frac{e^2B_0^2}{2mc^2 4}(x^2+y^2)\psi$$

$$\omega = \frac{eB_0}{mc} \rightarrow \omega' = \frac{\omega}{2}$$

$$H\psi = \frac{p^2}{2m}\psi + \omega' L_z\psi + \frac{1}{2}m(\omega')^2x^2\psi + \frac{1}{2}m(\omega')^2y^2\psi$$

again $\psi = X(x)\,Y(y)\,e^{ik_z z}$

z: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi = E_z\psi \Rightarrow E_z = \frac{\hbar^2}{2m}k_z^2$ and $\omega' L_z\psi = \omega'\hbar m_\ell = \frac{\omega\hbar}{2}m_\ell$

x: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}m(\omega')^2x^2\psi = E_x\psi \Rightarrow E_x = (n_x+\tfrac{1}{2})\hbar\omega' = (n_x+\tfrac{1}{2})\frac{\hbar\omega}{2}$

y: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m(\omega')^2y^2\psi = E_y\psi \Rightarrow E_y = (n_y+\tfrac{1}{2})\frac{\hbar\omega}{2}$

So

$$E = E_x + E_y + E_z = (n_x+\tfrac{1}{2})\frac{\hbar\omega}{2} + (n_y+\tfrac{1}{2})\frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2 + \frac{\omega\hbar}{2}m_\ell$$

from part (a) $E = (n_y+\tfrac{1}{2})\hbar\omega + \frac{\hbar^2}{2m}k_z^2$

if $n_y = 0$ $\quad E = \frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2$

in our case if $n_x = n_y = m_\ell = 0$

$$E = \frac{1}{2}\frac{\hbar\omega}{2} + \frac{1}{2}\frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2 = \frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2 \text{ (as above)}$$

So indeed the energy doesn't change (the degeneracy does though)