The Hamiltonian for a spinless charged particle in a magnetic field is

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2$$

where the magnetic field $\vec{B}$ is related to $\vec{A}$ via $\vec{B} = \nabla \times \vec{A}$. Let $\vec{A} = -B_0 y\,\hat{x}$ corresponding to $\vec{B} = B_0\hat{z}$

(a) Find the energy levels of the particle.

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2 = \frac{1}{2m}\left[p^2 - \frac{e}{c}\vec{A}\cdot\vec{p} - \frac{e}{c}\vec{p}\cdot\vec{A} + \frac{e^2}{c^2}A^2\right]$$

if this was a weak $\vec{B}$ field, we could ignore the A^2 term. But, it is not. So, using $\vec{A} = -B_0 y\,\hat{x}$, we get

$$H = \frac{1}{2m}\left[p^2 + \underbrace{\frac{2B_0 e}{c}\,y\,p_x}_{\text{since } [y,p_x]=0} + \frac{e^2}{c^2}B_0^2 y^2\right]$$

recall from $\frac{mv^2}{R} = qvB \underset{w=vR}{\Rightarrow} \omega = \frac{qB}{m} = \frac{-eB}{m}$. So, (for $c=1$)

$$H = \frac{p^2}{2m} - \omega\,y\,p_x + \frac{m\omega^2}{2}y^2$$

$$\text{let } u = \frac{-\omega p_x}{m\omega^2} + y$$

So,

$$\frac{1}{2}m\omega^2 u^2 - \frac{\omega^2 p_x^2}{2m\omega^2} = \frac{1}{2}m\omega^2 y^2 - \omega y\,p_x$$

Thus, our Hamiltonian is of the form

$$H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 u^2}_{H_0} - \underbrace{\frac{p_x^2}{2m}}_{H'}$$

we know that $E_0 = \langle H_0 \rangle = (n + \frac{1}{2})\,\omega = (n + \frac{1}{2})\,\frac{qB}{m}$

and we know that E' can be found from

$$-\frac{P_x^2}{2m}\psi = E'\psi \quad \Rightarrow \quad \frac{d^2\psi}{dx^2} + K_x^2\psi = 0 \quad , K_x^2 = 2mE$$

$$\Rightarrow \quad E = \frac{K_x^2}{2m}$$

Thus,

$$E = \langle H \rangle = \langle H_0 \rangle + \langle H' \rangle = \boxed{(n + \tfrac{1}{2})\frac{qB}{m} + \frac{K_x^2}{2m}}$$

(b) would the energy levels change if we choose $\vec{A}$ to be $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$?

note: $\nabla \cdot \vec{A} = \frac{B_0}{2}\left[\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right] = 0$

This is the Coulomb gauge. From gauge invariance, the energy levels will not change!