The Hamiltonian of a one-dimensional harmonic oscillator in dimensionless units ($m = \hbar = \omega = 1$) is

$$H = a^{\dagger} a + \frac{1}{2}$$

where $a = \frac{1}{\sqrt{2}}(x + ip)$, $a^{\dagger} = \frac{1}{\sqrt{2}}(x - ip)$. One of the unnormalized eigenfunctions of this Hamiltonian is given by the expression

$$\psi(x) = (2x^3 - 3x)\, e^{-x^2/2}$$

(a) what is the energy eigenvalue which corresponds to this wave function?

we know that $a^{\dagger} a |\psi_n(x)\rangle = n|\psi_n(x)\rangle$, where $a^{\dagger}a$ is the number operator

where $\underset{\substack{\uparrow \\ \text{unnormalized}}}{\psi(x)} \propto \psi_{n=?}(x)$

So, let's apply the lowering operator to $\psi(x)$

$$a\psi(x) = \frac{1}{\sqrt{2}}(x + ip)\left[(2x^3 - 3x)\, e^{-x^2/2}\right]$$

where $p \to -i\frac{d}{dx}$ in 1-d

That is,

$$a\psi(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\left[(2x^3 - 3x)\, e^{-x^2/2}\right]$$

$$= \frac{1}{\sqrt{2}}\left[(2x^4 - 3x^2)e^{-x^2/2} + (6x^2 - 3)e^{-x^2/2} - (2x^3 - 3x)\, x\, e^{-x^2/2}\right]$$

$$= \frac{1}{\sqrt{2}}\, e^{-x^2/2}\left[\underbrace{2x^4 - 2x^4}_{0} \underbrace{- 3x^2 + 3x^2}_{0} + 6x^2 - 3\right]$$

$$\Rightarrow a\psi(x) = \frac{3}{\sqrt{2}}(2x^2 - 1)\, e^{-x^2/2} \qquad (1)$$

Now, let's apply $a^\dagger$ to eq (1)

$$a^\dagger a \Psi(x) = \frac{3}{\sqrt{2}} \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx}\right)(2x^2-1)\, e^{-x^2/2}$$

$$= \frac{3}{2}\left[(2x^3-x)e^{-x^2/2} - 4x\, e^{-x^2/2} + (2x^2-1)\,x\, e^{-x^2/2} \right]$$

$$= \frac{3}{2} e^{-x^2/2} \left[2x^3 - x - 4x + 2x^3 - x \right]$$

$$= \frac{3}{2} e^{-x^2/2} (4x^3 - 6x)$$

$$\Rightarrow \quad a^\dagger a \Psi(x) = 3\, e^{-x^2/2}(2x^3 - 3x) = 3\, \Psi(x)$$

$$\boxed{\text{Thus, } n = 3}$$

(b) Find the two other (unnormalized) energy eigenfunctions which are closest in energy to this wave function.

So, we want to find the $n=2$ and $n=4$ wave functions, we get $n=2$ state by applying the lowering operator. From eq (1), we know this is

$$n=2 \qquad \boxed{a\Psi(x) \propto (2x^2-1)\, e^{-x^2/2}}$$

to get the $n=4$ state, we need to apply the raising operator $a^\dagger$, So, we have

$$a^\dagger \psi(x) = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\left(2x^3 - 3x\right)e^{-x^2/2}$$

$$= \frac{e^{-x^2/2}}{\sqrt{2}}\left(2x^4 - 3x^2 - (6x^2 - 3) + (2x^4 - 3x^2)\right)$$

$$= \frac{e^{-x^2/2}}{\sqrt{2}}\left(4x^4 - 12x^2 + 3\right)$$

$\therefore$ $n = 4$ $\boxed{a^\dagger \psi(x) \propto \left(4x^4 - 12x^2 + 3\right)e^{-x^2/2}}$