Spring 2002 #2 (p 1 of 2)

A system of two particles each with spin 1/2 is described by the Hamiltonian

$$H = A(S_{1z} + S_{2z}) + B\, S_1 \cdot S_2$$

Find all the energy levels of the Hamiltonian.

$$S = S_1 + S_2 \Rightarrow S^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2 \Rightarrow S_1 \cdot S_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2)$$

making the substitution for $S_1 \cdot S_2$ in our Hamiltonian, we get

$$H = A S_z + \frac{B}{2}(S^2 - S_1^2 - S_2^2)$$

where $S_z = S_{1z} + S_{2z}$

The possible values of S are $|S_1 - S_2| \le S \le |S_1 + S_2| \Rightarrow 0 \le S \le 1$.

So, S can be either 0 or 1.

Now, since $S_1 = \frac{1}{2}$ and $S_2 = \frac{1}{2}$ ($\hbar = 1$), the Hamiltonian is

$$H = A S_z + B\left(\frac{S^2}{2} - \frac{3}{4}\right)$$

where we used $S_i^2 = S_i(S_i + 1)$. Also note $S_z |m_z\rangle = m_z |m_z\rangle$

$S = 0$

we have the singlet state $\dfrac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(\frac{A S_z |\uparrow\downarrow\rangle - A S_z |\downarrow\uparrow\rangle}{\sqrt{2}} + \frac{B\left(\frac{S^2}{2} - \frac{3}{4}\right)|\uparrow\downarrow\rangle - B\left(\frac{S^2}{2} - \frac{3}{4}\right)|\uparrow\downarrow\rangle}{\sqrt{2}}\right)\Bigg|_{S=0}$$

note $S_z |\uparrow\downarrow\rangle = (+0 - 0)|\uparrow\downarrow\rangle = 0$ and same for $S_z |\downarrow\uparrow\rangle$

$$\Rightarrow E_{00} = \left(\frac{\langle\uparrow\downarrow| - \langle\downarrow\uparrow|}{\sqrt{2}}\right) H \left(\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left(\frac{-3B}{4} - \frac{3B}{4}\right)$$

$$\Rightarrow \quad \boxed{E_{00} = -\frac{3B}{4}}$$

$S = 1$

triplet states

$$|\uparrow\uparrow\rangle \quad \Rightarrow \quad E_{11} = \left[A\left(\tfrac{1}{2}+\tfrac{1}{2}\right) + B\left(\tfrac{2}{2}-\tfrac{3}{4}\right)\right]$$

$$\Rightarrow \boxed{E_{11} = A + \frac{B}{4}}$$

$$\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \quad \Rightarrow \quad E_{10} = \frac{1}{2}\left[A\cdot 0 + B\left(\tfrac{2}{2}-\tfrac{3}{4}\right)\right]$$

$$\Rightarrow \boxed{E_{10} = \frac{B}{4}}$$

$$|\downarrow\downarrow\rangle \quad \Rightarrow \quad E_{1,-1} = \left[A\left(-\tfrac{1}{2}-\tfrac{1}{2}\right) + B\left(\tfrac{1}{4}\right)\right]$$

$$\Rightarrow \boxed{E_{1,-1} = -A + \frac{B}{4}}$$