Spring 2002 #4

4. *Quantum Mechanics.*

Consider a mass m particle in one dimension moving in the potential

$$V(x) = V_0\left|\frac{x}{x_0}\right| \tag{1}$$

where V_0 and x_0 are constants.

Estimate the ground state energy of the particle.

Your score on this problem will be

$$\text{Your Score} = 20 \times e^{-\left(\frac{E-E_0}{E_0}\right)^2} \tag{2}$$

20 is the maximum score, E is your estimate, E_0 is the exact ground state energy, and E and E_0 are evaluated at $V_0 = \frac{\hbar^2}{m x_0^2} = 1\,\text{eV}$.

Lets pick trial wave function $\psi = e^{-\alpha x^2/2}$

$H = \frac{-1}{2m}\frac{\partial^2}{\partial x^2} + \frac{V_0}{X_0}|x|$ $\quad V_0, X_0$ are constants $\quad X_0 > 0$

$$E(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha\frac{x^2}{2}}\left(\frac{-1}{2m}\frac{\partial^2}{\partial x^2} + \frac{V_0}{X_0}|x|\right)e^{-\alpha\frac{x^2}{2}}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\alpha x^2}{2}}\left(-\tfrac{1}{2m}\alpha^2 x^2 e^{-\frac{\alpha x^2}{2}} + \frac{V_0}{X_0}|x| e^{-\alpha x^2/2}\right)dx$$

$$= \int_{-\infty}^{\infty}\frac{e^{-\alpha x^2}}{2m}\alpha^2 x^2\,dx + \int_{-\infty}^{0}\frac{V_0}{X_0}e^{-\alpha x^2}(-x)\,dx + \int_{0}^{\infty}\frac{V_0}{X_0}e^{-\alpha x^2}x\,dx$$

$$\int_{-\infty}^{\infty} x^2 e^{-a^2x^2}\,dx = \frac{1}{2a}\left(\frac{\pi}{a}\right)^{1/2} \qquad \int_0^{\infty} e^{-ax^2}x\,dx = \frac{1}{2a}$$

$u = -x$
$du = -dx$

$$\frac{V_0}{X_0}\int_{\infty}^{0} e^{-\alpha u^2}\,u\,(-)du \Rightarrow \frac{V_0}{X_0}\int_0^{\infty} e^{-\alpha u^2}\,u\,du \quad \text{set}$$

$$= -\frac{\partial^2}{2m}\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx \;\pm \frac{V_0}{X_0}\int_0^{\infty} e^{-\alpha x^2} x\,dx + \frac{V_0}{X_0}\int_0^{\infty} e^{-\alpha x^2} x\,dx$$

$$\frac{V_0}{X_0}\cdot\frac{1}{2\alpha}$$

$$= \frac{-\alpha^2}{2m}\cdot\frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{1/2} + \frac{V_0}{X_0}\frac{1}{2\alpha}$$

$$= \frac{-\alpha}{2m}\left(\frac{\pi}{\alpha}\right)^{1/2} + \frac{V_0}{X_0\alpha} = \frac{-1}{2m}\cdot(\alpha\pi)^{1/2} + \frac{V_0}{X_0\alpha}$$

$$\frac{\partial E}{\partial\alpha} = 0 = \frac{-1}{2m}\left(\frac{\pi}{\alpha}\right)^{1/2} - \frac{\alpha}{2m}\left(\tfrac{1}{2}\right)\left(\frac{\pi}{\alpha}\right)^{-1/2}(-\pi)\alpha^{-2} - \frac{V_0}{X_0\alpha^2}$$

$$= \frac{1}{\alpha}\frac{\pi}{4m}\left(\frac{\alpha}{\pi}\right)^{1/2} - \frac{V_0}{X_0\alpha^2} - \frac{1}{2m}\left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$0 = \frac{1}{4m\alpha}(\pi\alpha)^{1/2} - \frac{V_0}{X_0\alpha^2} - \frac{1}{2m}\left(\pi/\alpha\right)^{1/2}$$

$$= \frac{\alpha}{4m}(\pi\alpha)^{1/2} - \frac{V_0}{X_0} - \frac{\alpha}{2m}(\alpha\pi)^{1/2}$$

$$\frac{\alpha}{4m}(\pi\alpha)^{1/2} = \frac{-V_0}{X_0} \qquad \frac{\alpha^3\pi}{(4m)^2} = \left(\frac{V_0}{X_0}\right)^2 \qquad \left(\frac{V_0}{X_0}\right)^2\frac{(4m)^2}{\pi} = \alpha^3$$

$$\alpha = \left(\frac{V_0}{X_0}\right)^{1/3}\frac{(4m)^{2/3}}{\pi^{1/3}}$$

$$E_0 \approx \frac{-1}{2m}\left(\frac{V_0}{X_0}\right)^{1/6}\frac{(4m)^{1/3}}{\pi^{1/6}}\pi^{1/2} + \frac{V_0}{X_0 2}\frac{\pi^{1/3}}{(4m)^{2/3}}\left(\frac{X_0}{V_0}\right)^{1/3}$$

$$= -\frac{1}{2m}\left(\frac{V_0}{x_0}\right)^{1/6} \pi^{1/3} (4m)^{1/3} + \left(\frac{V_0^2}{x_0 5}\right)^{1/3} \frac{\pi^{1/3}}{(4m)^{2/3}}$$

$$E_{min} = \pi^{1/3} (4m)^{1/3} \left(\frac{V_0^{2/3}}{4m\ x_0^{5/3}} - \frac{V_0^{1/6}}{2m x_0^{1/6}}\right)$$

$$= \frac{\pi^{1/3} (4m)^{1/3}}{m x_0^2} \left(\frac{V_0^{2/3}\ x_0^{1/3}}{4} - \frac{V_0^{1/6} x_0^{11/6}}{2}\right)$$