Consider a mass m particle in one dimension moving in the potential

$$V(x) = V_0 \left| \frac{x}{x_0} \right|$$

where V_0 and x_0 are constants. Estimate the ground state of the particle.

E_0 is the exact ground state energy. $V_0 = \frac{\hbar^2}{m x_0^2} = 1\,eV$.

Use variational method. ... for other approaches see Bertrand's solution. Choose trial function

$$\psi(x) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$$

The Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x) = -\frac{1}{2m}\frac{d^2}{dx^2} + V_0\left|\frac{x}{x_0}\right|$$

Now find $\langle H \rangle$.

$$\langle H \rangle = \frac{1}{2m}\langle p^2 \rangle + \frac{V_0}{|x_0|}\langle |x| \rangle$$

$$\frac{1}{2m}\langle p^2 \rangle = -\frac{1}{2m}\sqrt{\frac{2a}{\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\frac{d^2}{dx^2}e^{-ax^2}\,dx = -\frac{1}{2m}\sqrt{\frac{2a}{\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\frac{d}{dx}(-2ax)e^{-ax^2}\,dx$$

$$= -\frac{1}{2m}\sqrt{\frac{2a}{\pi}}\int_{-\infty}^{\infty}(-2a)e^{-2ax^2}\,dx - \frac{1}{2m}\sqrt{\frac{2a}{\pi}}\int_{-\infty}^{\infty}4a^2x^2e^{-2ax^2}\,dx$$

$$\Rightarrow \frac{1}{2m}\langle p^2 \rangle = \frac{a}{m}\sqrt{\frac{2a}{\pi}}\sqrt{\frac{\pi}{2a}} - \frac{2a^2}{m}\sqrt{\frac{2a}{\pi}}\frac{\sqrt{\pi}}{2(2a)^{3/2}} = \frac{a}{m} - \frac{a^2}{m}\frac{1}{2a} = \boxed{\frac{a}{2m}} \quad (1)$$

$$\frac{V_0}{|x_0|}\langle |x| \rangle = \frac{V_0}{|x_0|}\int_{-\infty}^{\infty}\underbrace{e^{-ax^2}|x|e^{-ax^2}}_{\text{even function}}\,dx = \frac{2V_0}{|x_0|}\int_0^{\infty} x\,e^{-2ax^2}\,dx$$

let $u = x^2 \Rightarrow du = 2x\,dx$

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$$\Rightarrow \frac{V_0}{|X_0|}\langle |X|\rangle = \frac{2V_0}{|X_0|}\int_0^\infty \frac{1}{2} e^{-2au}\,du = \frac{V_0}{|X_0|}\left(-\frac{1}{2a}\right)\left[e^{-2au}\right]_0^\infty = \frac{V_0}{2a|X_0|}$$

Thus, combining eqs (1) and (2) we get

$$\langle H\rangle = \frac{a}{2m} + \frac{V_0}{2a|X_0|}$$

now find the extremum.

$$0 = \frac{\partial\langle H\rangle}{\partial a} = \frac{1}{2m} - \frac{V_0}{2a^2|X_0|} \xrightarrow{\text{solving for } a} \frac{V_0}{2|X_0|} = \frac{1}{2m}a^2$$

$$\therefore \quad a = \sqrt{\frac{mV_0}{|X_0|}}$$

then

$$E = \langle H(a)\rangle = \frac{1}{2m}\sqrt{\frac{mV_0}{|X_0|}} + \frac{V_0}{2|X_0|}\sqrt{\frac{|X_0|}{mV_0}} = \frac{1}{2}\sqrt{\frac{V_0}{m|X_0|}} + \frac{1}{2}\sqrt{\frac{V_0}{m|X_0|}}$$

$$= \sqrt{\frac{V_0}{\underbrace{m|X_0|}_{1/V_0}}} = \sqrt{V_0^2} = \boxed{V_0}$$

so, my score is $20 \times e^{-\left(\frac{V_0 - V_0}{V_0}\right)^2} = 20$