Charge on a circle: A small bead with charge e and mass m is confined to move on a circular ring in the x-y plane with radius r. A weak, uniform electric field of intensity E_0 pointing in the positive x direction is turned on.

(a) What are the eigenfunctions and energy eigenvalues for $E_0 = 0$? What are the degeneracies? (See Fall 1996 #11)

When $E_0 = 0$, we just have

$$H = \frac{p^2}{2m} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{P_\phi^2}{2mr^2 \sin^2\theta}$$

since the charge is constrained to a circular ring in the x-y plane, we know that

$$P_r = 0 = P_\theta$$

and $\theta = \frac{\pi}{2}$. So, we have

$$H = \frac{P_\phi^2}{2mr^2} = \frac{L_z^2}{2mr^2}$$

with $\hbar = 1$, the Schrödinger eq. is

$$-\frac{1}{2mr^2}\frac{d^2}{d\phi^2}\psi = E\psi \Rightarrow \frac{d^2}{d\phi^2}\psi + n^2\psi = 0, \quad n^2 = 2mr^2 E$$

the solutions to this ΔE are

$$\psi(x) \propto e^{\pm in\phi}$$

since these functions repeat every 2π, n must be an integer

So, the energy levels are

$$\boxed{E_n = \frac{n^2}{2mr^2}} \quad , n = 0, \pm 1, \pm 2, \pm 3$$

there is 2-fold degeneracy for every n except $n = 0$. This comes from the energy being the same no matter if going clockwise or counter clockwise

(b) For $E_0 \neq 0$, show that the electric field operator has vanishing matrix elements between degenerates of the unperturbed Hamitonian.

the perturbation is given by

$$H' = -qEx = eEx$$

the diagonal elements vanish.

The reason is that x is odd and diagonal elements have the same function so they have the same parity. So, we have

$$E_0^{(1)} = eE \langle \psi(x) \,|\, x \,|\, \psi(x) \rangle = 0$$

odd

same parity

(c) Find an approximation for the energy levels which includes the first non-trivial term containing E_0.

from part (b) we know that 1st order perturbation vanishes. So, try 2nd order.

$$E_0^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m \,|\, H' \,|\, \psi_n \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

where $E_n^{(0)} - E_m^{(0)} = \dfrac{1}{2Mr^2}(n^2 - m^2)$

change expression from mass to avoid confusion with index.

and

$$\langle \psi_m | H' | \psi_n \rangle = eE\int_{-\infty}^{\infty} e^{-im\phi}\, x\, e^{in\phi}\, d\phi = eEr\int_{-\infty}^{\infty} e^{-im\phi} \cos\phi\, e^{in\phi}\, d\phi$$

since $\cos\phi = \frac{1}{2}\left(e^{-i\phi} + e^{i\phi}\right)$, we have

$$\langle \psi_m | H' | \psi_n \rangle = \frac{eEr}{2} \int_{-\infty}^{\infty} e^{-i\phi(m-n)} \left(e^{-i\phi} + e^{i\phi}\right) d\phi$$

$$= \frac{eEr}{2} \int_{-\infty}^{\infty} \left[e^{-i\phi(m-n-1)} + e^{-i\phi(m-n+1)} \right] d\phi$$

$$= \frac{eEr}{2} \left[\delta_{m-n-1,0} + \delta_{m-n+1,0} \right]$$

So, only two values of m survive

$$m = n \pm 1$$

for $m = n+1$, $\langle \psi_m | H' | \psi_n \rangle = \frac{1}{2} eEr$

$m = n-1$, $\langle \psi_m | H' | \psi_n \rangle = \frac{1}{2} eEr$

So,

$$E_0^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m | H' | \psi_n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{|\frac{1}{2} eEr|^2}{\frac{1}{2Mr^2}} \left[\frac{1}{n^2 - (n+1)^2} + \frac{1}{n^2 - (n-1)^2} \right]$$

$$= \frac{1}{4} e^2 E^2 r^2 (2Mr^2) \underbrace{\left[\frac{1}{n^2 - n^2 - 2n - 1} + \frac{1}{n^2 - n^2 - 1 + 2n} \right]}_{= -\frac{1}{2n+1} + \frac{1}{2n-1} = \frac{-2n+1+2n+1}{4n^2-1} = \frac{2}{4n^2-1}}$$

Thus,

$$\boxed{E_0^{(2)} = \frac{e^2 E^2 r^4 M}{4n^2 - 1}}$$