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charge on a circle: A small bead with charge e and mass m is confined to move on a circular ring in the x - y plane with radius r . A weak, uniform electric field of intensity E_0 pointing in the positive x direction is turned on.

(a) What are the eigenfunctions and energy eigenvalues for $E_0 = 0$? What are the degeneracies? (see Fall 1996 #11)

When $E_0 = 0$, we just have

$$H = \frac{p^2}{2m} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\theta}$$

since the charge is constrained to a circular ring in the x - y plane, we know that

$$p_r = 0 = p_\theta$$

and $\theta = \frac{\pi}{2}$, so, we have

$$H = \frac{p_\phi^2}{2mr^2} = \frac{L_z^2}{2mr^2}$$

with $\hbar = 1$, the Schrödinger eq. is

$$-\frac{1}{2mr^2} \frac{d^2}{d\phi^2} \psi = E \psi \Rightarrow \frac{d^2}{d\phi^2} \psi + n^2 \psi = 0, \quad n^2 = 2mr^2 E$$

the solutions to this ΔE are

$$\psi(\phi) \propto e^{\pm i n \phi}$$

since these functions repeat every 2π , n must be an integer

So, the energy levels are

$$\boxed{E_n = \frac{n^2}{2mr^2}} \quad , n = 0, \pm 1, \pm 2, \pm 3$$

there is 2-fold degeneracy for every n except $n=0$. This comes from the energy being the same no matter if going clockwise or counter clockwise

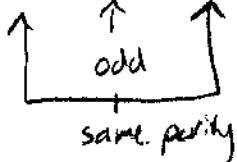
(b) For $E_0 \neq 0$, show that the electric field operator has vanishing matrix elements between degenerates of the unperturbed Hamiltonian.

the perturbation is given by

$$H' = -q \vec{E} x = e E x$$

the diagonal elements vanish.

The reason is that x is odd and diagonal elements have the same function so they have the same parity. So, we have

$$E_0^{(1)} = e E \langle \psi(\xi) | x | \psi(\xi) \rangle = 0$$


(c) Find an approximation for the energy levels which includes the first non-trivial term containing E_0 .

from part (b) we know that 1st order perturbation vanishes, so, try 2nd order.

$$E_0^{(2)} = \sum_{n \neq 0} \frac{|\langle \psi_n | H' | \psi_0 \rangle|^2}{E_0^{(0)} - E_n^{(0)}}$$

$$\text{where } E_n^{(0)} - E_0^{(0)} = \frac{1}{2M r^2} (n^2 - 0^2)$$

↑
change expression from mass to avoid confusion with index.

and

$$\langle \psi_m | H' | \psi_n \rangle = e E \int_{-\infty}^{\infty} e^{-im\phi} x e^{in\phi} d\phi = e E r \int_{-\infty}^{\infty} e^{-im\phi} \cos\phi e^{in\phi} d\phi$$

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since $\cos \phi = \frac{1}{2} (e^{-i\phi} + e^{i\phi})$, we have

$$\begin{aligned}\langle \psi_m | H' | \psi_n \rangle &= \frac{eEr}{2} \int_{-\infty}^{\infty} e^{-i\phi(m-n)} (e^{-i\phi} + e^{i\phi}) d\phi \\ &= \frac{eEr}{2} \int_{-\infty}^{\infty} \left[e^{-i\phi(m-n-1)} + e^{-i\phi(m-n+1)} \right] d\phi \\ &= \frac{eEr}{2} \left[\delta_{m-n-1,0} + \delta_{m-n+1,0} \right]\end{aligned}$$

So, only two values of m survive

$$m = n \pm 1$$

$$\begin{aligned}\text{for } m = n+1, \quad \langle \psi_m | H' | \psi_n \rangle &= \frac{1}{2} eEr \\ m = n-1, \quad \langle \psi_m | H' | \psi_n \rangle &= \frac{1}{2} eEr\end{aligned}$$

So,

$$E_0^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m | H' | \psi_n \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{|\frac{1}{2} eEr|^2}{\frac{1}{2Mr^2}} \left[\frac{1}{n^2 - (n+1)^2} + \frac{1}{n^2 - (n-1)^2} \right]$$

$$\begin{aligned}&= \frac{1}{4} e^2 E^2 r^2 (2Mr^2) \left[\frac{1}{n^2 - n^2 - 2n - 1} + \frac{1}{n^2 - n^2 - 1 + 2n} \right] \\ &= \frac{-1}{2n+1} + \frac{1}{2n-1} = \frac{-2n+1 + 2n+1}{4n^2 - 1} = \frac{2}{4n^2 - 1}\end{aligned}$$

Thus,

$$E_0^{(2)} = \frac{e^2 E^2 r^4 M}{4n^2 - 1}$$