S'02 E.M. #8

We start out with the wave equation for $\vec{E}$:

$$\nabla^2 \vec{E} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{E}}{\partial t} = 0 \qquad \text{but} \quad \vec{J} = \sigma \vec{E}$$

so

$$\nabla^2 \vec{J} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial \vec{J}}{\partial t} = 0$$

we can assume the solution $\vec{J} = J_x(y)\, e^{i\omega t}$, and as we are dealing with only one dimension (y):

$$\frac{d^2}{dy^2} J(y) - \frac{4\pi\sigma\mu}{c^2} i\omega J(y) = 0$$

now define $\tau^2 = i\frac{4\pi\sigma\mu\omega}{c^2}$, as $\sqrt{i} = \frac{\sqrt{2}}{2}(1+i)$

$$\tau = (1+i)\sqrt{\frac{2\pi\sigma\mu\omega}{c}} = \frac{1+i}{\delta} \quad \text{with} \quad \delta = \frac{c}{\sqrt{2\pi\sigma\mu\omega}} = \frac{c}{\beta}$$

so we now have:

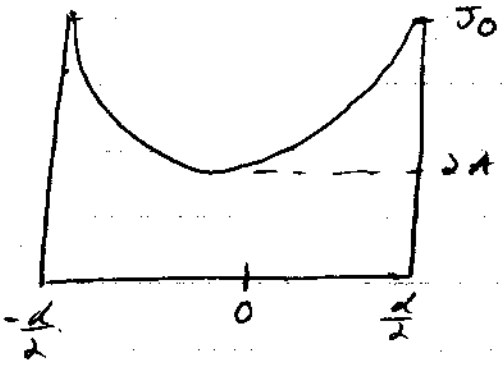
$$\frac{d^2}{dy^2} J(y) - \tau^2 J(y) = 0 \quad \Rightarrow \quad J(y) = Ae^{-\tau y} + Be^{\tau y}$$

by planar symmetry this can be written as:

$$\boxed{J(y) = A\left(e^{-\tau y} + e^{\tau y}\right)}$$

as
$$\left.\begin{aligned} J\left(y = -\tfrac{d}{2}\right) &= J_0 = A\left(e^{\tau d/2} + e^{-\tau d/2}\right) \\ J\left(y = +\tfrac{d}{2}\right) &= J_0 = A\left(e^{-\tau d/2} + e^{\tau d/2}\right) \end{aligned}\right\} \text{ they are the same}$$

so the plot of $|J(y)|$ is:



Now for the phase difference:

$$J(0) = 2A; \quad J\left(y = \tfrac{+d}{2}\right) = A\left(e^{-\tau d/2} + e^{\tau d/2}\right) = J\left(y = -d/2\right)$$

taking the ratio:

$$\frac{J(y = \pm d/2)}{J(0)} = \frac{e^{-\tau d/2} + e^{\tau d/2}}{2}$$

but $\tau = \frac{1+i}{\delta} = \frac{(1+i)}{c}\beta$

so $= \frac{1}{2}\left(e^{-\frac{(1+i)}{c}\beta d/2} + e^{\frac{(1+i)}{c}\beta d/2}\right)$

for $\beta d \gg 1$ the above $\simeq \frac{1}{2} e^{\frac{(1+i)}{c}\beta d/2}$